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SHAPE FACTORS FOR FIRST FORBIDDEN NON-UNIQUE BETA TRANSITIONS

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Abstract: New approximations are developed for the shape factors in non-unique beta decay. Simplifications are possible because the combination of the so-called forbidden beta-decay functions $(L_\nu M_\nu)^{\frac{1}{2}} \pm N_\nu$ (upper sign for negatons, lower sign for positons) is very small as compared to the functions themselves.

1. Introduction

The theoretical formulae for the shape of continuous beta spectra as given by the Fermi theory [1]) are rather unwieldy, especially in the case of non-unique transitions where the shape is determined by the accidental values of the nuclear matrix elements. This is even true if the beta-decay interaction contains only S, T and P terms, as assumed until recently [1,2]), or only V and A interactions as would follow from the latest experiments [19,20]) as well as from the theory of Feynman and Gell-Mann [3]).

A knowledge of the spectral shapes is important in measurements of internal conversion coefficients of gamma rays by the method of comparing internal conversion line intensities with the intensity of a continuous beta spectrum. Many measured beta spectra are significantly distorted for momenta less than about half the maximum momentum on account of back-scattering in the source and in the source backing and also by scattering in different parts of the beta-ray spectrometer. The lower part of the spectrum has therefore to be constructed by extrapolation from the higher energy measurements, using the theoretical spectral shape. The spectral shape should be known with a precision of a few percent, if the conversion coefficients are to be determined with a slightly higher accuracy. We have therefore tried to find a useful approximation for the spectral shapes in non-unique transitions.

2. Results of Beta Decay Theory

The theoretical energy distribution in an n^{th} forbidden electron spectrum is

$$N(p)\mathrm{d}p = (g^2/2\pi^3)p^2q^2F_0S_n/(n!)^2\mathrm{d}p; \qquad (2.1)$$

p and q are the electron and neutrino momenta and F_0 the Fermi function. Formulae for the shape factor S_n are given by Greuling [4]) and Pursey [5]) and (more correct for pseudoscalar interaction [8])) by Konopinski and Uhlenbeck [6]) and Smith [7]). Rearranging their terms we can write

$$S_n = S_{n,n} + S_{n,n+1} + \delta_{n,1} S_{10} \tag{2.2}$$

where, for a combination of vector and axial vector interactions,

$$S_{n,n} = \sum_{\nu=0}^{n-1} (n-\nu)(2n-2\nu+1) B_{n\nu} q^{2(n-\nu-1)} \Big[M_\nu (v_n \pm a_n)^2 + 2N_\nu (v_n \pm a_n) \Big(\xi v_{na} - \frac{v_n \mp a_n}{2n-2\nu+1} q \Big) + L_\nu \Big(\xi v_{na} - \frac{v_n \mp a_n}{2n-2\nu+1} q \Big)^2 \Big], \tag{2.3}$$

$$S_{n,n+1} = \sum_{\nu=0}^{n} B_{n\nu} q^{2(n-\nu)} L_\nu \Big[\frac{a_{nu}^2}{(n+1)^2} + \frac{(n+1)}{(2\nu+1)(2n-2\nu+1)} \Big(v_n \pm \frac{n-2\nu}{n+1} a_n \Big)^2 \Big],$$

$$S_{10} = M_0 a_{10}^2 - 2N_0 a_{10} (\xi a_{a1} + \tfrac{1}{3} a_{10} q) + L_0 (\xi a_{a1} + \tfrac{1}{3} a_{10} q)^2.$$

The upper signs refer to negatons, the lower ones to positons The parameter ξ is defined

$$\xi = \hbar \alpha Z / 2mcr \approx 0.42 Z A^{-\frac{1}{3}}. \tag{2.4}$$

In most cases ξ is a rather large number: $\xi \approx \pm 10$. The Greuling constant $B_{n\nu}$ can be written in the following symmetric way:

$$B_{n\nu} = \frac{1}{\nu!(n-\nu)!} \frac{(2\nu+1)!!}{(2n-2\nu+1)!!}. \tag{2.5}$$

The parameters v and a are combinations of interaction constants and matrix elements explained in a symbolic way in table 1. In a general parity non-conserving theory, their squares and products should be defined, e.g.,

$$v_0^2 = (|C_V|^2 + |C'_V|^2) \Big| \int 1 \Big|^2,$$

$$2 v_0 a_{0u} = \mathrm{Re}\, (C_V C_A + C'_V C'_A) \Big[\int 1 \int \sigma^* + \mathrm{c.c.} \Big].$$

Then the relation $(v_0 a_{0u})^2 = v_0^2 a_{0a}^2$ generally does not hold so that the formulae (2.3) have to be used with some caution. The normal relations between squares and products are, however, valid in the two-component neutrino theory with time reversal and in the Feynman–Gell-Mann theory.

Formulae for a combination of scalar, tensor and pseudoscalar interactions are obtained by the following substitutions:

$$q \to -q, \qquad v_n \to s_n, \qquad a_n \to t_n, \qquad v_{na} \to \pm t_{na},$$
$$a_{nu} \to t_{nu}, \qquad a_{10} \to t_{10}, \qquad a_{a1} \to \pm p_{a1}.$$

The parameters s, t and p (column 2 in table 1) are defined in the same way as the corresponding v and a's in the first column except that a factor β has to be added under the integral signs.

TABLE 1

Definition of parameters in formula (2.3)

Our notation		Usual Notation			Selection Rules		
VA	(STP)	$n=0$	$n=1$	$n=2$	J	J_i+J_f	$\pi_i\pi_f$
v_n	(s_n)	$C_V\int 1$	$C_V\int ir$	$C_V R_{i_1\ldots i_n}$	$\leqq n$	$\geqq n$	$(-1)^n$
a_n	(t_n)		$C_A\int \sigma\times r$	$C_A T_{i_1\ldots i_n}$	$\leqq n$	$\geqq n$	$(-1)^n$
ξv_{n0}	(ξt_{n0})		$C_V\int \alpha$	$C_A A_{i_1\ldots i_n}$	$\leqq n$	$\geqq n$	$(-1)^n$
a_{n0}	(t_{n0})	$C_A\int \sigma$	$C_A B_{ij}$	$C_A B_{i_1\ldots i_{n+1}}$	$\leqq n+1$	$\geqq n+1$	$(-1)^n$
a_{10}	(t_{10})		$C_A\int \frac{1}{i}\sigma\cdot r$		0		-1
ξa_{21}	(ξp_{21})		$C_A\int \gamma_5$		0		-1

A combination of all five interactions would, in addition, give rise to Fierz terms containing combinations of matrix elements $\mathrm{Re}(C_{VA}C_{STP}+C'_{VA}C'_{STP})$. The results of measurements of longitudinal polarization of electrons [9]) indicate that (if VA and STP interactions occur in comparable amounts) $C'_{VA}\approx +C_{VA}$ and $C'_{STP}\approx -C_{STP}$ (in agreement with the prediction of Feynman and Gell-Mann [7])); the Fierz terms should therefore be small. Spectrum shape measurements [10]) point to the same conclusion; a similar and even more accurate result is obtained from measurements of electron capture to positon emission ratios on ^{22}Na [11]), ^{18}F [12]), ^{48}V and ^{57}Ni [13]). We shall therefore disregard these terms.

Transition rates for electron capture in the X_σ shell can be computed from the combination of eqs. (2.1—3) if the functions L_ν, N_ν and M_ν, multiplied by $2p^2F_0$, are replaced by functions defined as follows:

X_{2n-1} shell: $L_{n-1}=n\pi R^{2-2n}f^2$ $\quad$ X_{2n} shell: $L_{n-1}=n\pi R^{-2n}g^2$

$N_{n-1}=n\pi R^{1-2n}fg$ $\quad$ $N_{n-1}=n\pi R^{-1-2n}fg$

$M_{n-1}=n\pi R^{-2n}g^2$; $\quad$ $M_{n-1}=n\pi R^{-2-2n}f^2$.

L_ν, N_ν and M_ν vanish for $\nu\neq n-1$; R is short for $mcr/\hbar$, r being the radius of the nucleus; on the right hand sides, the factor n has to be replaced by half the number of electrons in the shell under consideration, if this shell is not completely filled. Data about the functions f^2 and g^2 in the K and L shells can be found in a report by Brysk and Rose [14]).

3. The New Approximation

The functions L, M and N have the following properties:

$$M_\nu\approx L_\nu\xi^2/(\nu+1)^2, \tag{3.1}$$

$$N_\nu\approx \mp(L_\nu M_\nu)^{\frac{1}{2}}\approx -\xi L_\nu/(\nu+1). \tag{3.2}$$

The upper sign again applies to negaton emission and electron capture, the lower to positon emission. Eq. (3.2) is exact for electron capture, and sur-

prisingly accurate for electron emission in all practically occurring cases, as is proved as follows. We define

$$O_\nu = R^{-1-2\nu}(f_{\nu+1}f_{-\nu-1}-g_{\nu+1}g_{-\nu-1}).$$

Then $L_\nu M_\nu = N_\nu^2+O_\nu^2$. Using second order approximations for the electron wave functions we find

$$O_\nu = p/(2\nu+3)W. \tag{3.2'}$$

Taking all these data together, we derive

$$\frac{(L_\nu M_\nu)^{\frac{1}{2}}\pm N_\nu}{N_\nu} = \frac{(\nu+1)^2}{2\xi^2(2\nu+3)^2}\left(\frac{P}{W}\right)^2 \lesssim 0.02\ \%.$$

In fact we found from the tables by Rose *et al.*[15]) that the difference between $\mp N_\nu$ and $(L_\nu M_\nu)^{\frac{1}{2}}$ is at most 0.3 % for negatons and $Z = 10$, and considerably smaller for larger Z. For positons a complication arises due to the fact that N_ν has a zero (and M_ν a minimum). This zero occurs, however, for large energies (5 MeV for $Z = 10$, 8 MeV for $Z = 20$, and still higher energies for higher Z). For all known first forbidden positon transitions the error introduced by replacing $\mp N_\nu$ by $(L_\nu M_\nu)^{\frac{1}{2}}$ would again be smaller than 0.3 %.

Combining eqs. (2.2) and (3.2) one obtains an expression for the part of $S_{n,n}$ between parentheses which is nearly a square. $S_{n,n}$ can therefore be written

$$\begin{aligned} S_{n,n} &= \sum_{\nu=0}^{n-1}(n-\nu)(2n-2\nu+1)B_{n\nu}\xi^2 q^{2(n-\nu-1)}L_\nu(C_{n\nu}+b_{n\nu}q)^2+P_{n\nu}, \\ C_{n\nu} &= v_{n\mathrm{a}}-(v_n\pm a_n)\frac{1}{\xi}\left(\frac{M_\nu}{L_\nu}\right)^{\frac{1}{2}}, \qquad b_{n\nu} = \frac{1}{\xi}\frac{v_n\mp a_n}{2n-2\nu+1}. \end{aligned} \tag{3.3}$$

The quantity S_{10} can be simplified in a similar way. The quantities $P_{n\nu}$, and a fortiori their influence on the energy dependence of the correction factors S_n, are small compared with the other terms except when

$$v_{n\mathrm{a}}(v_n\pm a_n)/C_{n\nu}^2 \gg 1. \tag{3.4}$$

This can only arise as a result of an accidental cancellation of matrix elements, which is in general improbable. A similar relation holds for S_{10}.

4. First Forbidden Transitions

We consider first forbidden transitions with spin change and with normal *ft* values, meaning that no accidental cancellation of matrix elements occurs. Then, if the decay energy is not too high, only the shape factor $S_{1,1}$ is important; and in eq. (3.3) we may take $P_{1,0} = 0$. The parameter $(M_0/L_0)^{\frac{1}{2}}$ in this formula is a nearly linear function of the electron energy (compare fig. 1) and if we write

$$(M_0/L_0)^{\frac{1}{2}}/\xi = \pm d+eq/\xi, \tag{4.1}$$

then $d \approx 1$ and $e \approx 0.5$. The function L_0 is also a nearly linear function of the electron energy (compare fig. 2):

$$L_0 \propto (1-fq/\xi); \tag{4.2}$$

f is small for low Z and approaches 0.5 for high Z. Combining these approximations with eq. (3.3) we obtain

$$S \propto (1+gq)^2, \tag{4.3}$$

where

$$g = \frac{1}{3\xi}\left[\frac{v_n(1-3e) \mp a_n(1+3e)}{v_{nn}-d(a_n \pm v_n)} - \frac{3}{2}f\right]. \tag{4.4}$$

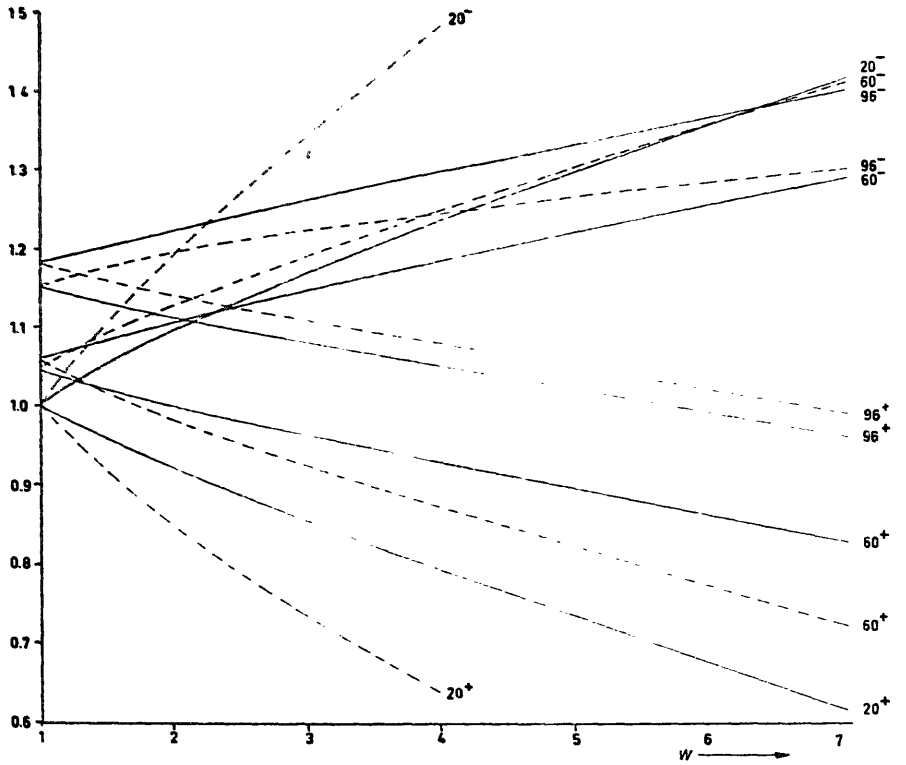

Fig. 1. The functions M_0/ξ^2(dotted) and $(M_0/L_0\xi^2)^{\frac{1}{2}}$ (drawn) as a function of the relativistic electron energy. The Z values are indicated; the attached signs refer to positons (+) or negatons (−).

The term S_{10} which should be added for transitions without spin change can be written in a similar way; the two terms can in most cases be combined without significant loss of precision into one shape factor having the same form as eq. (4.3).

In view of the above considerations we propose that first forbidden transitions should be analysed by plotting $(N/p^2F_0)^{\frac{1}{2}}/W_0-W)$ as a function

of the electron energy; this plot should yield a straight line with a slope g. The best value for W_0, being that value for which the plot is most nearly linear, can be obtained most easily by trial and error. (In allowed transition the same plot should give a straight line with a slope $g = f/2\xi$.)

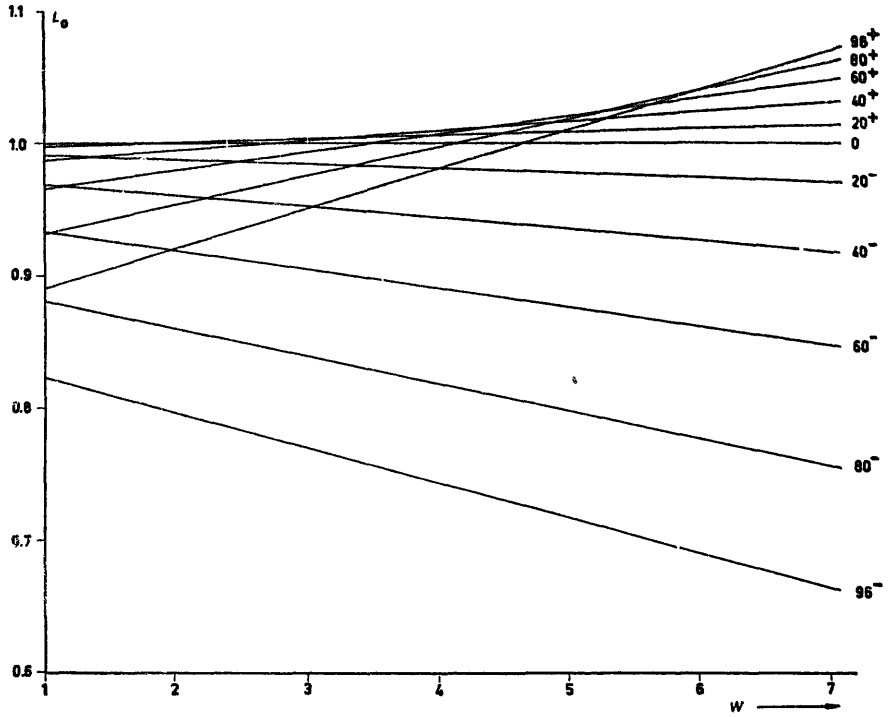

Fig. 2. L_0 for positons (+) and negatons (−) as a function of the electron energy for different values of Z. As in fig. 1, the function L_0 has been taken from the work of Rose *et al.* [15]).

5. Some Examples

We shall show the results of such an analysis on a measurement of Au^{198}. This isotope has recently been remeasured in this laboratory [16]). Fig. 3 shows the Fermi plot of one of these measurements. The deviations from a straight line are very small, in agreement with earlier observations mostly using at least ten times thicker sources; they are, however, definitely systematic. As a result we get a "forbidden plot" (fig. 4) which can be represented by a straight line above 200 keV. Fig. 5 shows the influence of this method of extrapolating the β-spectrum; the area under the curve extrapolated in the way discussed above is 2.3 % larger than that obtained in the usual way. This difference could not be neglected, since it was intended to use this area for a determination of a conversion coefficient with an error of at most a few percent. The deviations below 200 keV are probably due to scattering, as will be discussed elsewhere.

In the case of ^{198}Au the Fermi plot was found to be curved away from the energy axis, meaning that the constant g in eq. (4.6) was positive (+0.058). Porter *et al.*[7]), studying the component in the 940 keV β-spectrum of ^{186}Re

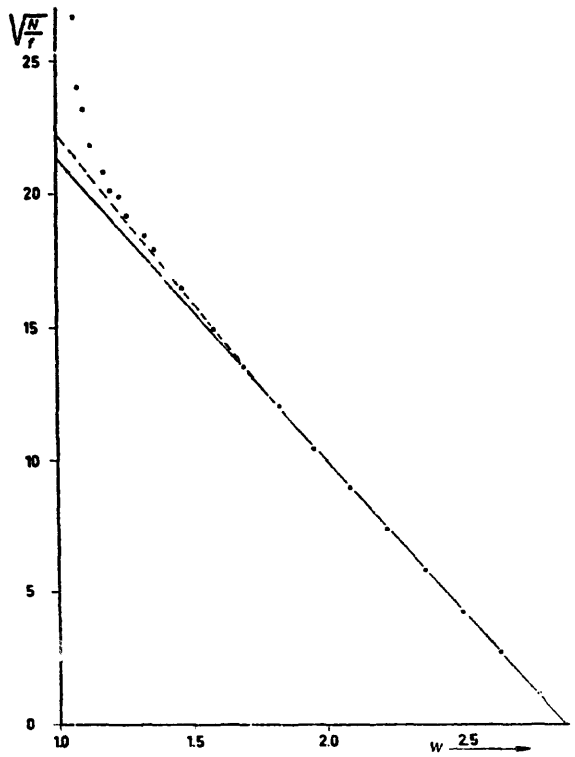

Fig. 3. Fermi plot of a measurement of the main component in the continuous beta spectrum of ^{198}Au. The dotted line is recomputed from the "forbidden plot" (fig. 4). The function f, taken from Nat. Bur. Standards, Applied Mathematics, Series 13, is proportional to p^2F as defined in section 1; it has been corrected for screening.

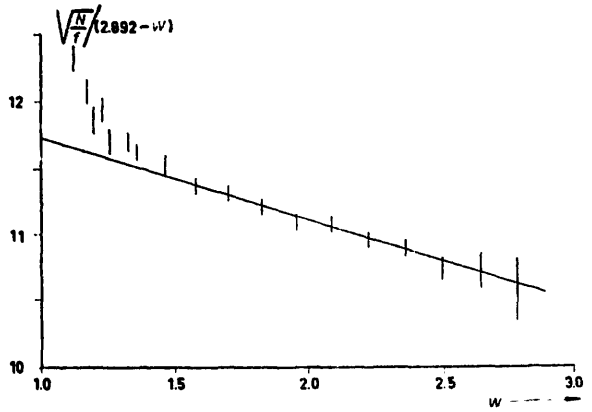

Fig. 4. First "forbidden plot" of the data given in fig. 3.

leading to the first excited state in ^{186}Os, found here a Fermi plot curved towards the energy axis; the shape factor is proportional to $1-(0.061\pm 0.015)q^2$. In this case the constants defined in section 4 are $d = 1.174$,

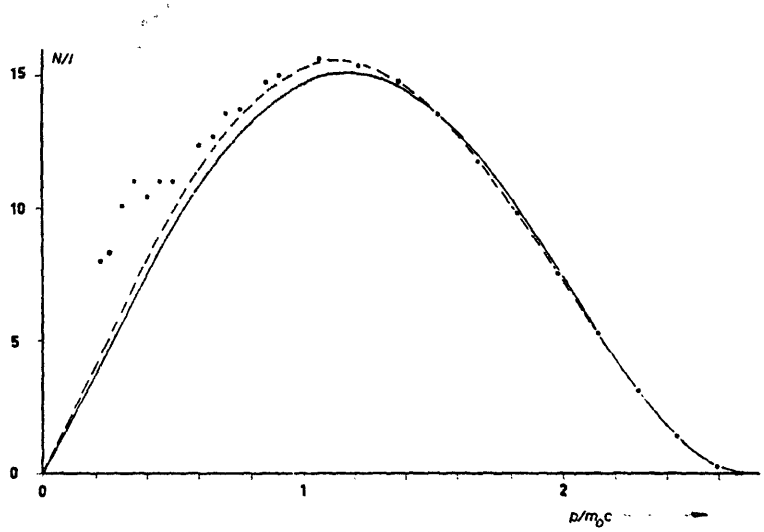

Fig. 5. Beta spectrum of ^{198}Au. The drawn line is the extrapolation from the straight line Fermi plot in fig. 3; the dotted line is the extrapolation using the analysis of fig. 4.

$e = 0.52$, $f = 0.29$, $\xi = 13.27$ (found from an analysis of the beta decay functions given by Rose *et al.*[15]). Introducing these values in eq. (4.7) (the transition under consideration is a $J = 1$ transition) we find the following relation between the matrix elements playing a role in this decay:

$$v_{n\mathrm{a}} - 2.5a_n - 1.5v_n = 0.$$

The matrix element ratios themselves cannot be determined from this type of experiment except under the following conditions:

A) Cancellation must occur, so that the above approximation is not valid. Usually $S_{1,2}$ is then comparable to $S_{1,1}$ so that the matrix element $a_{n\mathrm{u}}$ becomes important, unless

B) the transition under consideration is a 1—0 transition.

Both conditions are fulfilled in the decay of RaE; this explains that in RaE Wu [2]) could give separate values $t_{1\mathrm{a}}/t_1 = 1.51$ and $s_1/t_1 = 0.22$. It is easily checked that with these values S_{12} becomes comparable to S_{11}.

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