IX. OPTICAL PULSE PROPAGATION

THE ELECTROMAGNETIC NONLINEAR SCHRÖDINGER EQUATION:

We begin our discussion of optical pulse propagation[35] with a derivation of the nonlinear Schrödinger (NLS) equation. To that end, we recall Equations [VII-23] and [VII-23] from the early lecture set entitled *Nonlinear Optics I -- i.e.*

$$\nabla^2\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right) + \left(\omega^2/c^2\varepsilon_0\right)\bar{\bar{\varepsilon}}(\omega)\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -\mu_0\,\omega^2\,\vec{\mathbf{P}}^{(\mathrm{NL})}(\vec{\mathbf{r}},\omega) \qquad [\text{ IX-1 }]$$

$$\vec{\nabla}\cdot\left[\bar{\bar{\varepsilon}}(\omega)\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right] = -\vec{\nabla}\cdot\vec{\mathbf{P}}^{(\mathrm{NL})}(\vec{\mathbf{r}},\omega) \qquad [\text{ IX-2 }]$$

In this treatment we will confine our attention to wave propagation in **uniform, isotropic optical materials** -- *viz.*, glass fibers. For such materials, we can write

$$\vec{\mathbf{P}}^{(\mathrm{NL})}(\vec{\mathbf{r}},\omega) = \varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega)\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad [\text{ IX-3 }]$$

where $\varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega) = \frac{3}{4}\varepsilon_0\,\chi^{(3)}_{xxxx}\left|\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right|^2$ and, thus, Equation [IX-1] simplifies to

$$\nabla^2\,\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) + \left(k_0^2/\varepsilon_0\right)\left[\varepsilon(\omega) + \varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega)\right]\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = 0 \qquad [\text{ IX-4 }]$$

where $k_0 = \omega/c$.[36]

To proceed, postulate that this nonlinear Helmholtz equation can be treated by ***separation of variables*** methods. In particular, we are looking for a time-localized solution (a

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[35] An excellent reference on this subject is Govind P. Agrawal's *Nonlinear Fiber Optics*, Academic Press (1989) ISBN 0-12-045140-9.

[36] In this simplification, we have taken $\vec{\nabla}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = -\left[\varepsilon(\omega) - \varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega)\right]^{-1}\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\cdot\vec{\nabla}\varepsilon_{\mathrm{NL}}(\vec{\mathbf{r}},\omega) \approx 0$.

pulse) with a relatively narrow frequency spectrum (or "group" of frequencies centered on a frequency ω_{ctr}. Thus, we assume a seoaration of variables solution

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \mathrm{F}(x,y)\ \vec{\mathbf{G}}(z,\omega-\omega_{\text{ctr}})\ \exp\left(-\,i\,\beta_{\text{ctr}}\,z\right) \qquad \text{[IX-5]}$$

where β_{ctr} is a wave or propagation number to be associated with ω_{ctr} and, thus, Equation [IX-4] becomes

$$\left\{\frac{\partial^2 \mathrm{F}}{\partial x^2}+\frac{\partial^2 \mathrm{F}}{\partial y^2}+\left[k^2-\bar{\beta}^2\right]\mathrm{F}\right\}\vec{\mathbf{G}}+\left\{\frac{\partial^2\vec{\mathbf{G}}}{\partial z^2}-i\,2\beta_{\text{ctr}}\frac{\partial\vec{\mathbf{G}}}{\partial z}+\left[\bar{\beta}^2-\beta_{\text{ctr}}^2\right]\vec{\mathbf{G}}\right\}\mathrm{F}=0 \qquad \text{[IX-6]}$$

where $k = k_0\sqrt{\left[\varepsilon(\omega)+\varepsilon_{\text{NL}}(\vec{\mathbf{r}},\omega)\right]/\varepsilon_0}$. In the linear problem $\bar{\beta}^2$ would be the "separation constant," but in this case we will need a bit more elaboration. Nevertheless, we shall assume that we can find a set of functions $\mathrm{F}(x,y)$ and values $\bar{\beta}^2$ that satisfy the equation

$$\frac{\partial^2 \mathrm{F}}{\partial x^2}+\frac{\partial^2 \mathrm{F}}{\partial y^2}+\left[k^2-\bar{\beta}^2\right]\mathrm{F}=0 \qquad \text{[IX-7a]}$$

so that

$$\frac{\partial^2\vec{\mathbf{G}}}{\partial z^2}-i\,2\beta_{\text{ctr}}\frac{\partial\vec{\mathbf{G}}}{\partial z}+\left[\bar{\beta}^2-\beta_{\text{ctr}}^2\right]\vec{\mathbf{G}}=0 \qquad \text{[IX-7b]}$$

To use perturbation theory, we first reduce Equation [IX-7a] to a solvable linear problem by writing

$$\begin{aligned} k^2 &= k_0^2\left[\varepsilon(\omega)+\varepsilon_{\text{NL}}(\vec{\mathbf{r}},\omega)\right]/\varepsilon_0 = \left[n(\omega)+\Delta\,n(\omega)\right]^2 k_0^2 \\ &\approx \left[n^2(\omega)+2\,n(\omega)\,\Delta\,n(\omega)\right]\ k_0^2 \end{aligned} \qquad \text{[IX-8a]}$$

$$\bar{\beta}^2=\left[\beta+\Delta\beta\right]^2\approx\beta^2+2\beta\,\Delta\beta \qquad \text{[IX-8b]}$$

where $\Delta n(\omega) = \dfrac{\varepsilon_{NL}(\vec{\mathbf{r}},\omega)}{2\,n(\omega)} = \dfrac{3}{8}\,\dfrac{\chi^{(3)}_{xxxx}(\omega)}{n(\omega)}\left|\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\right|^2$. Thus, to first order we need to solve the linear equation

$$\frac{\partial^2 \mathrm{F}}{\partial x^2} + \frac{\partial^2 \mathrm{F}}{\partial y^2} + \left[n^2\, k_0^2 - \beta^2\right]\mathrm{F} = 0\;. \qquad [\text{ IX-9 }]$$

which taken together with appropriate boundary conditions defines the linear eigenvalue problem for propagation in the medium where the functions F are the eigenfunctions and the values β are the eigenfunctions. In the previous lecture set -- *i.e.,* VIII. Guided Waves in Planar Structures -- we found a set of eigenfunctions and eigenvalues appropriate to the dielectric-slab guidewave propagation

Following an earlier discussion, we now presume that $\left|\dfrac{\partial^2 \vec{\mathbf{G}}}{\partial z^2}\right| << \left|2\beta_{ctr}\,\dfrac{\partial \vec{\mathbf{G}}}{\partial z}\right|$ -- *i.e.*, we take the ***slowly vary amplitude*** or ***paraxial*** approximation -- so that Equation [IX-7b] reduces to

$$-i\,2\beta_{ctr}\,\frac{\partial \vec{\mathbf{G}}}{\partial z} + \left[\overline{\beta}^{\,2} - \beta_{ctr}^2\right]\vec{\mathbf{G}} = 0 \qquad [\text{ IX-10a }]$$

where $\overline{\beta}^{\,2} - \beta_{ctr}^2 \approx \beta^2 - \beta_{ctr}^2 + 2\,\beta\,\Delta\beta$ and, thus,

$$-i\,2\beta_{ctr}\,\frac{\partial \vec{\mathbf{G}}}{\partial z} + \left[\beta^2 - \beta_{ctr}^2\right]\vec{\mathbf{G}} + 2\,\beta\,\Delta\beta\,\vec{\mathbf{G}} = 0 \qquad [\text{ IX-10b }]$$

For equation [IX-7a] to be completely satisfied in first order, we must have

$$\Delta\beta = \frac{n\,k_0^2}{\beta}\,\frac{\iint \Delta n\,\left|\mathrm{F}\left(x,y\right)\right|^2\,dx\,dy}{\iint \left|\mathrm{F}\left(x,y\right)\right|^2\,dx\,dy}$$

$$= \frac{3\,k_0^2}{8\,\beta}\,\chi_{xxxx}^{(3)}\left|\vec{\mathbf{G}}\left(z,\omega-\omega_{\mathrm{ctr}}\right)\right|^2\,\frac{\iint \left|\mathrm{F}\left(x,y\right)\right|^4\,dxdy}{\iint \left|\mathrm{F}\left(x,y\right)\right|^2\,dxdy} \qquad \text{[IX-11]}$$

Since we are assuming that propagating pulse has a relatively narrow frequency spectrum, it is reasonable to use a Taylor expansion around ω_{ctr} for $\beta\left(\omega\right)$ -- *viz.*

$$\beta\left(\omega\right) \approx \beta_{\mathrm{ctr}} + \left(\omega-\omega_{\mathrm{ctr}}\right)\beta_1 + \frac{1}{2!}\left(\omega-\omega_{\mathrm{ctr}}\right)^2\beta_2 + \frac{1}{3!}\left(\omega-\omega_{\mathrm{ctr}}\right)^3\beta_3 + \ldots \qquad \text{[IX-12]}$$

where $\beta_\ell = \left.\dfrac{d^\ell\beta\left(\omega\right)}{d\omega^\ell}\right|_{\omega=\omega_c}$.

Near the **dispersion minimum** in glass fibers (*i.e.* $\lambda \approx 1.3-1.6\,\mu m$) we may, to very good approximation, stop with the quadratic term and write

$$\beta^2 - \beta_{\mathrm{ctr}}^2 \approx 2\beta_{\mathrm{ctr}}\left[\beta-\beta_{\mathrm{ctr}}\right] = 2\,\beta_{\mathrm{ctr}}\left[\left(\omega-\omega_{\mathrm{ctr}}\right)\beta_1 + \frac{1}{2!}\left(\omega-\omega_{\mathrm{ctr}}\right)^2\beta_2\right]. \qquad \text{[IX-13]}$$

In this approximation, Equation [IX-10] becomes

$$\frac{\partial\vec{\mathbf{G}}}{\partial z} + i\left[\left(\omega-\omega_{\mathrm{ctr}}\right)\beta_1 + \frac{1}{2!}\left(\omega-\omega_{\mathrm{ctr}}\right)^2\beta_2\right]\vec{\mathbf{G}} + \frac{i\,\beta}{\beta_{\mathrm{ctr}}}\,\Delta\beta\,\vec{\mathbf{G}} = 0\ . \qquad \text{[IX-14a]}$$

If we take

$$\gamma = \frac{\beta}{\beta_{\mathrm{ctr}}}\,\Delta\beta = \frac{3\,k_0^2}{8\beta_{\mathrm{ctr}}}\,\chi_{xxxx}^{(3)}\,\frac{\iint \left|\mathrm{F}\left(x,y\right)\right|^4\,dxdy}{\iint \left|\mathrm{F}\left(x,y\right)\right|^2\,dxdy} \qquad \text{[IX-15]}$$

Equation [IX-10] in the **frequency domain** reduces to

$$\frac{\partial \vec{\mathbf{G}}}{\partial z} + i \left[(\omega - \omega_c)\, \beta_1 + \frac{1}{2!} (\omega - \omega_c)^2\, \beta_2 \right] \vec{\mathbf{G}} = -i\, \gamma \left| \vec{\mathbf{G}} \right|^2 \vec{\mathbf{G}} \ . \qquad \text{[IX-14b]}$$

which implies the following **time domain equation** for the pulse envelope:

$$\frac{\partial \vec{\mathbf{G}}(z,t)}{\partial z} + \beta_1 \frac{\partial \vec{\mathbf{G}}(z,t)}{\partial t} - \frac{i}{2} \beta_2 \frac{\partial^2 \vec{\mathbf{G}}(z,t)}{\partial t^2} + \frac{\alpha}{2} \vec{\mathbf{G}}(z,t) = -i\, \gamma \left| \vec{\mathbf{G}}(z,t) \right|^2 \vec{\mathbf{G}}(z,t) \qquad \text{[IX-16]}$$

The last term on the left hand side has been added to incorporate the effects of various possible loss mechanisms.

Next we transform into a coordinate system which moves with the "group" -- what might be called the ***surfer's coordinates*** of the pulse -- *i.e.*, $\{t,z\}$ $\{\tau,\varsigma\}$ where $\tau = t - t_0 - \beta_1 (z - z_0)$ and $\varsigma = z - z_0 = z - \beta_1^{-1} t_0$. In term of these surfer's coordinates, Equation [IX-16] becomes[37]

$$\frac{\partial \vec{\mathbf{G}}(\varsigma,\tau)}{\partial \varsigma} = -\frac{\alpha}{2} \vec{\mathbf{G}}(\varsigma,\tau) + \frac{i}{2} \beta_2 \frac{\partial^2 \vec{\mathbf{G}}(\varsigma,\tau)}{\partial \tau^2} - i\, \gamma \left| \vec{\mathbf{G}}(\varsigma,\tau) \right|^2 \vec{\mathbf{G}}(\varsigma,\tau) \qquad \text{[IX-17]}$$

Obviously, if we omit all of the terms on the right hand side of this equation so that $\dfrac{\partial \vec{\mathbf{G}}(\varsigma,\tau)}{\partial \varsigma} = 0$, we would have the ideal situation wherein a **pulse of any shape propagates forever without changing shape at a velocity** $v_g = \beta_1^{-1}$ --the group

[37] Since

$$\frac{\partial}{\partial z} \vec{\mathbf{G}}(z,t) = \frac{\partial}{\partial \varsigma} \vec{\mathbf{G}}(\varsigma,\tau) \frac{\partial \varsigma}{\partial z} + \frac{\partial}{\partial \tau} \vec{\mathbf{G}}(\varsigma,\tau) \frac{\partial \tau}{\partial z} = \frac{\partial}{\partial \varsigma} \vec{\mathbf{G}}(\varsigma,\tau)\,(1) + \frac{\partial}{\partial \tau} \vec{\mathbf{G}}(\varsigma,\tau) \left(-v_g^{-1}\right)$$

$$\frac{\partial}{\partial t} \vec{\mathbf{G}}(z,t) = \frac{\partial}{\partial \varsigma} \vec{\mathbf{G}}(\varsigma,\tau) \frac{\partial \varsigma}{\partial t} + \frac{\partial}{\partial \tau} \vec{\mathbf{G}}(\varsigma,\tau) \frac{\partial \tau}{\partial t} = \frac{\partial}{\partial \varsigma} \vec{\mathbf{G}}(\varsigma,\tau)\,(0) + \frac{\partial}{\partial \tau} \vec{\mathbf{G}}(\varsigma,\tau)\,(1) \ .$$

velocity. In treating the less-than-ideal situation, we will initially neglect the effects of the loss (first) term and confine our attention to the competing effects of the **dispersion** (second) and **nonlinear** (third) term . We cast the lossless version of Equation [IX-16] into a normalized, standard form by introducing

$$u\left(Z,T\right)=\frac{\vec{\mathbf{G}}\left(\varsigma,\tau\right)}{\sqrt{P_0}}\,,\qquad Z=\frac{\varsigma}{L_{\mathrm{D}}}=\varsigma\frac{|\,\beta_2\,|}{\Delta\tau^2}\,,\qquad T=\sqrt{2}\,\frac{\tau}{\Delta\,\tau}$$

where $\Delta\tau$ is the width of the pulse and P_0 is its peak power.

Thus, we, at last, obtain the standard form of the **nonlinear Schrödinger (NLS) equation**

$$i\,\frac{\partial u\left(Z,T\right)}{\partial Z}-\operatorname{sgn}\left(\beta_2\right)\frac{\partial^2 u\left(Z,T\right)}{\partial T^2}+N^2\left|u\left(Z,T\right)\right|^2 u\left(Z,T\right)=0 \qquad \text{[IX-18]}$$

where $N^2=L_{\mathrm{D}}/L_{\mathrm{NL}}=\gamma\,P_0\,\Delta\tau^2/|\,\beta_2\,|$.

PULSE SOLUTIONS OF LINEAR SCHRÖDINGER EQUATION

If in Equation [IX-10] we set $\vec{\mathbf{G}}\left(z,t\right)\quad U\left(z,t\right)$ and neglect the loss and nonlinear terms, we see that $\mathrm{U}\left(z,t\right)$ satisfies the following differential equation:

$$\frac{\partial}{\partial z}\,\mathrm{U}\left(z,t\right)+v_g^{-1}\frac{\partial}{\partial t}\,\mathrm{U}\left(z,t\right)=i\,b\,\frac{\partial^2}{\partial t^2}\,\mathrm{U}\left(z,t\right) \qquad \text{[IX-19]}$$

For **minimal dispersion** -- *viz.* if $b=\frac{1}{2!}\left.\frac{d^2\beta\left(\omega\right)}{d\omega^2}\right|_{\omega=\omega_0}\approx 0$ -- Equation [IX-19] becomes

$$\frac{\partial}{\partial z}\,\mathrm{U}\left(z,t\right)+v_g^{-1}\frac{\partial}{\partial t}\,\mathrm{U}\left(z,t\right)=0 \qquad \text{[IX-20]}$$

which is the basic wave equation for **minimally dispersive media** with the general solution

$$\mathrm{U}(z,t) = \mathrm{U}(z - v_g\, t) \qquad \text{[IX-21]}$$

where $v_g = \left[\left. \frac{d\,\beta(\omega)}{d\omega} \right|_{\omega=\omega_0} \right]^{-1}$ is the **group velocity** of the pulse.

When limitation to **first order dispersion** is an adequate approximation, Equation [IX-19] expressed in surfer's coordinates becomes

$$b\,\frac{\partial^2}{\partial\tau^2}\mathrm{U}(\varsigma,\tau) + i\,\frac{\partial}{\partial\varsigma}\mathrm{U}(\varsigma,\tau) = 0 \qquad \text{[IX-22]}$$

where $b = \beta_2/2$.

Amazingly, this pulse dispersion equation is the, so called, parabolic equation that we saw earlier in connection with beam propagation -- *viz.* the paraxial wave propagation equation.

Solution of Pulse Dispersion Equation

For convenience, we restate here Equation [IX-22 } the first-order pulse dispersion equation -- *viz.*

$$b\,\frac{\partial^2}{\partial\tau^2}\mathrm{U}(\varsigma,\tau) + i\,\frac{\partial}{\partial\varsigma}\mathrm{U}(\varsigma,\tau) = 0 \ .$$

Let us write a Fourier transform for this modulation in terms of "surfer time" -- *i.e.*

$$U(\varsigma,\tau) = \int_{-\infty}^{+\infty} U(\varsigma,\omega)\ \exp(i\,\omega\,\tau)\ d\omega \qquad \text{[IX-23a]}$$

where

$$U(\varsigma,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(\varsigma,\tau)\ \exp(-i\,\omega\,\tau)\ d\tau\ . \qquad \text{[IX-23b]}$$

Thus, the pulse dispersion equation reduces to an ordinary differential equation -- *viz*.

$$i\,\frac{d}{d\varsigma}\,U(\varsigma,\omega) = b\,\omega^2\ U(\varsigma,\omega) \qquad \text{[IX-24]}$$

for the Fourier transform and thus we have the simple solution

$$U(\varsigma,\omega) = U(0,\omega)\ \exp\left(-i\,b\,\omega^2\,\varsigma\right)\ . \qquad \text{[IX-25]}$$

Thus, we see that the dispersion changes the phase of **each spectral component** of the pulse by an amount that depends on the frequency and the propagated distance. The general solution may be written as

$$U(\varsigma,\tau) = \int_{-\infty}^{+\infty} U(0,\omega)\ \exp\left[i\left(\omega\,\tau - b\,\omega^2\,\varsigma\right)\right]\ d\omega \qquad \text{[IX-26a]}$$

where

$$U(0,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(0,t-t_0)\ \exp\left[-i\,\omega\left(t-t_0\right)\right]\ d\left(t-t_0\right)\ . \qquad \text{[IX-26b]}$$

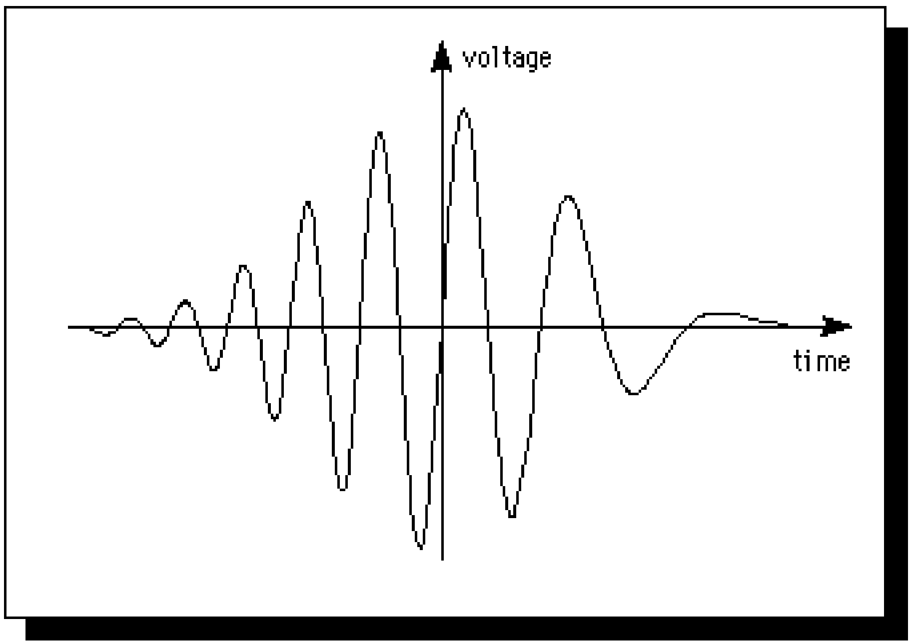
Let us suppose that we have a "chirped" Gaussian at $\varsigma = 0$ -- *i.e.*

$$U(0, t - t_0) = \exp\left[-\left(\frac{1 - iC}{2}\right)\left(\frac{t - t_0}{\tau_0}\right)^2\right] \qquad \text{[IX-27a]}$$

so that

$$U(0, \omega) = \sqrt{\frac{\tau_0^2}{2\pi(1 - iC)}}\, \exp\left[-\frac{\tau_0^2\, \omega^2}{2(1 - iC)}\right] \qquad \text{[IX-27b]}$$

where $C > 0$ characterizes an "up-chirp" and $C < 0$ a "down-chirp" pulse.



A Gaussian pulse with a frequency "down-chirp"

Inverting the transform, we see that

$$U(\varsigma,\tau)=\left[\frac{\tau_0^2}{\tau_0^2+i\,2\,b\,\varsigma(1-iC)}\right]\exp\left\{-\frac{(1-iC)\,\tau^2}{2\left[\tau_0^2+i\,2\,b\,\varsigma\,(1-iC)\right]}\right\} \qquad \text{[IX-28a]}$$

or rationalizing

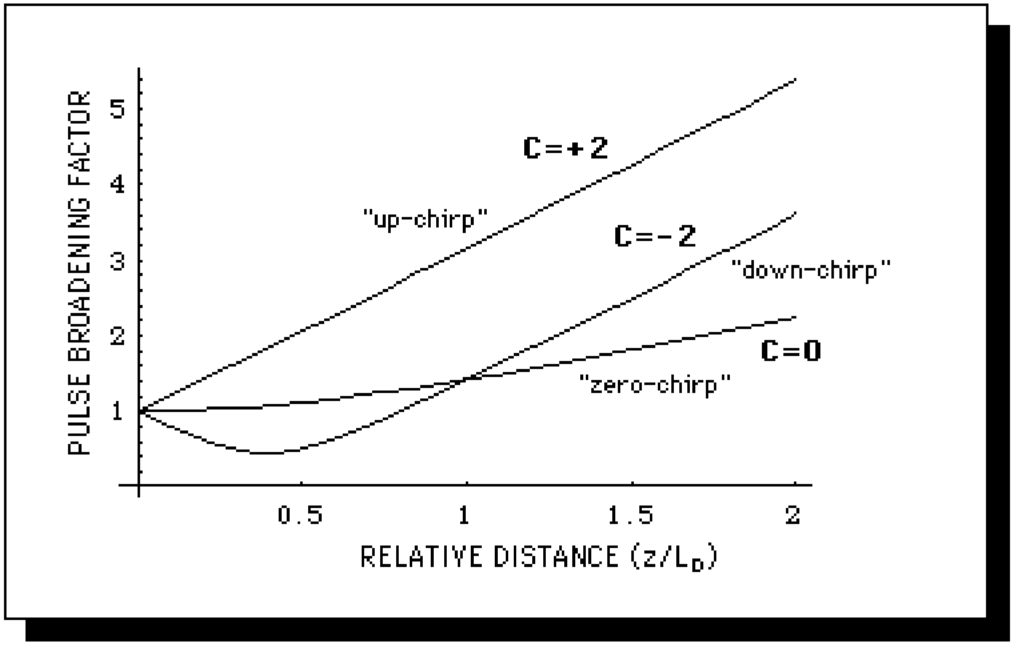
$$U(\varsigma,\tau)=\left[\frac{\tau_0^2}{\tau_0^2+i\,2\,b\,(1-iC)}\right]\exp\left\{-\frac{\tau^2}{2\,\tau_0^2\left[\left(1+\frac{2\,b\,C\varsigma}{\tau_0^2}\right)^2+\left(1+\frac{2\,b\,\varsigma}{\tau_0^2}\right)^2\right]}\right\}$$

$$\times\exp\left\{\frac{i\left[C+\frac{2\,b\,\varsigma}{\tau_0^2}\left(1+C^2\right)\right]\tau^2}{2\,\tau_0^2\left[\left(1+\frac{2\,b\,C\varsigma}{\tau_0^2}\right)^2+\left(1+\frac{2\,b\,\varsigma}{\tau_0^2}\right)^2\right]}\right\} \qquad \text{[IX-28b]}$$

Hence, the **pulse width broadening factor** at a given position z is given by

$$\frac{\tau(\varsigma)}{\tau_0}=\sqrt{\left(1+\frac{2\,bC\,\varsigma}{\tau_0^2}\right)^2+\left(\frac{2\,b\,\varsigma}{\tau_0^2}\right)^2}=\sqrt{\left(1+\frac{C\varsigma}{L_D}\right)^2+\left(\frac{\varsigma}{L_D}\right)^2} \qquad \text{[IX-29]}$$

where $L_D=\tau_0^2/2\,b$.



The spatial evolution of the pulse width of chirped Gaussian pulse

(For "normal dispersion" -- *i.e.* $\dfrac{\partial^2\beta}{\partial\omega^2} > 0$ **)**

Pulse Solutions of "Negligible Dispersion" Nonlinear Schrödinger Equation

If group velocity dispersion can be neglected, Equation [IX-17] reduces to

$$\frac{\partial \vec{\mathbf{G}}(\varsigma,\tau)}{\partial\varsigma} = -\frac{\alpha}{2}\,\vec{\mathbf{G}}(\varsigma,\tau) - i\,\gamma\left|\vec{\mathbf{G}}(\varsigma,\tau)\right|^2\vec{\mathbf{G}}(\varsigma,\tau) \qquad \text{[IX-30]}$$

and if we take $\vec{\mathbf{G}}(\varsigma,\tau) \quad U(\varsigma,\tau)\sqrt{P_0}\,\exp(-\alpha\varsigma/2)$ we obtain

$$\frac{\partial U(\varsigma,\tau)}{\partial\varsigma} = -i\,\mathrm{L}_{\mathrm{NL}}^{-1}\,\exp(-\alpha\varsigma)\left|U(\varsigma,\tau)\right|^2 U(\varsigma,\tau) \qquad \text{[IX-31]}$$

where the characteristic nonlinear length is given by $\mathrm{L}_{\mathrm{NL}} = (\gamma\, P_0)^{-1}$. A solution to this equation is readily obtain in the form

$$U(\varsigma,\tau) = U(0,\tau)\,\exp\left[i\,\Phi_{\mathrm{NL}}(\varsigma,\tau)\right] \qquad \text{[IX-32a]}$$

where

$$\begin{aligned}\Phi_{\mathrm{NL}}(\varsigma,\tau) &= -\left[1-\exp(-\alpha\varsigma)\right]\left[\alpha\,\mathrm{L}_{\mathrm{NL}}\right]^{-1}\exp(-\alpha\varsigma)\left|U(0,\tau)\right|^2 \\ &= -\left[\varsigma_{\mathrm{eff}}/\mathrm{L}_{\mathrm{NL}}\right]\left|U(0,\tau)\right|^2\end{aligned} \qquad \text{[IV-32b]}$$

and

$$\varsigma_{\mathrm{eff}} = \left[1-\exp(-\alpha\varsigma)\right]\alpha^{-1} \qquad \text{[IX-32c]}$$

This interesting result show that so called “self-phase modulation” or SPM gives rise to an **intensity dependent phase shifted or chirped pulse which remains constant in shape** as it propagates. The instantaneous optical frequency shift is given by

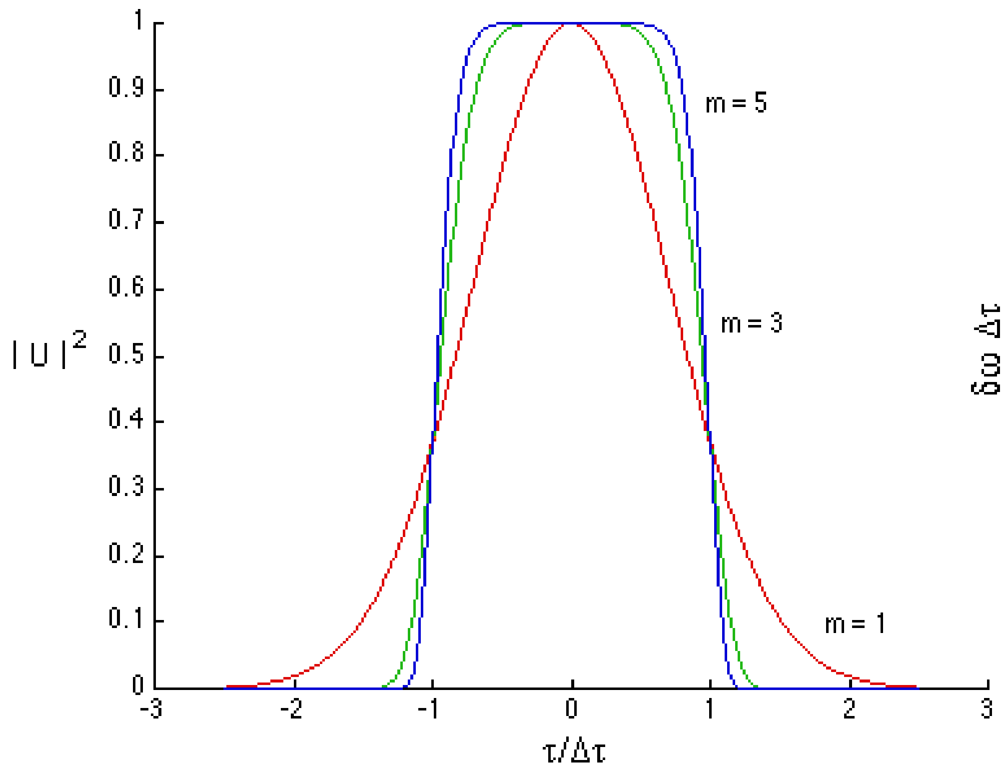
$$\delta\omega(\tau) = \frac{\partial\Phi_{\mathrm{NL}}(\varsigma,\tau)}{\partial\,\tau} = -\left[\varsigma_{\mathrm{eff}}/\mathrm{L}_{\mathrm{NL}}\right]\frac{\partial\left|U(0,\tau)\right|^2}{\partial\,\tau} \qquad \text{[IV-33]}$$

Note that the pulse spectrum is “red-shifted” on the leading edge of a pulse and “blue-shifted” on the trailing edge of the pulse. If we suppose the initial pulse to be a super-Gaussian of mth-order -- *i.e.*,

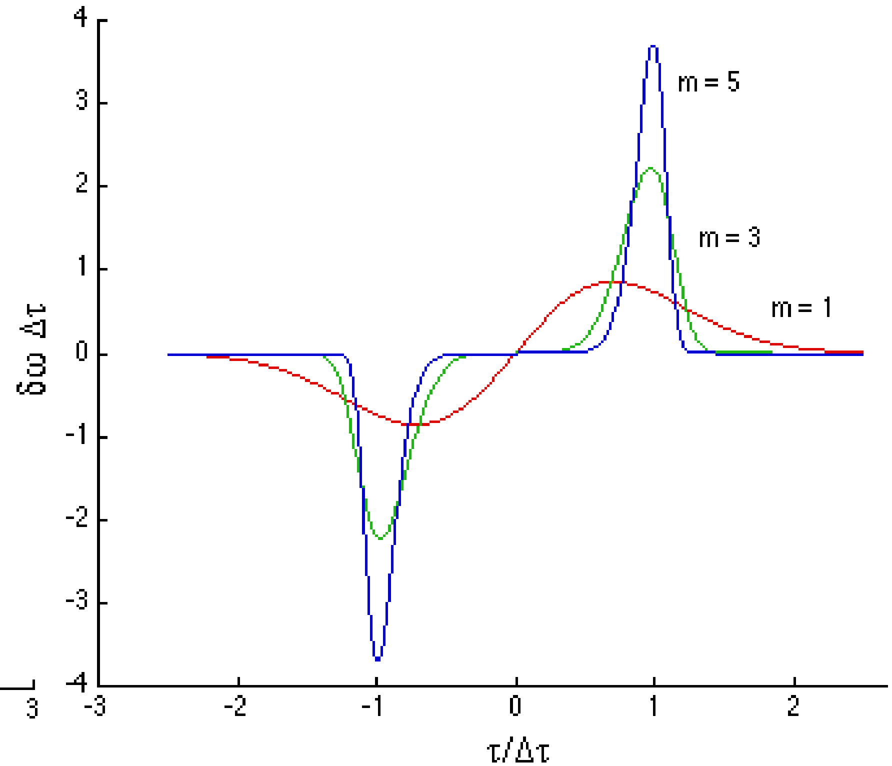
$$U_{\mathrm{m}}(0,\tau) = \exp\left[-\frac{1+iC}{2}\left(\frac{\tau}{\Delta\tau}\right)^{2\mathrm{m}}\right] \qquad \text{[IV-34]}$$

-- then the instantaneous optical frequency shift would be given by

$$\delta\omega(\tau) = \left(\frac{\varsigma_{\mathrm{eff}}}{\mathrm{L}_{\mathrm{NL}}}\right)\left(\frac{2\mathrm{m}}{\Delta\tau}\right)\left(\frac{\tau}{\Delta\tau}\right)^{2\mathrm{m}-1}\exp\left[-\left(\frac{\tau}{\Delta\,\tau}\right)^{2\mathrm{m}}\right] \qquad \text{[IV-35]}$$



Super-Gaussian Envelope Shapes



SPM Induced Frequency Chirp

A Brief Discussion of Solitons

We return briefly to Equation [IX-18] -- the standard form of the **nonlinear Schrödinger (NLS) equation** -- *i.e.*,

$$i\,\frac{\partial u(Z,T)}{\partial Z} - \mathrm{sgn}(\beta_2)\,\frac{\partial^2 u(Z,T)}{\partial T^2} + N^2\left|u(Z,T)\right|^2 u(Z,T) = 0 \qquad \text{[IX-18]}$$

where $u\left(Z,T\right)=\dfrac{\vec{\mathbf{G}}\left(\varsigma,\tau\right)}{\sqrt{P_0}}$, $Z=\dfrac{\varsigma}{L_{\mathrm{D}}}=\varsigma\dfrac{|\beta_2|}{\Delta\tau^2}$, $T=\sqrt{2}\,\dfrac{\tau}{\Delta\,\tau}$ and $N^2=L_{\mathrm{D}}/L_{\mathrm{NL}}=\gamma P_0\,\Delta\tau^2/|\beta_2|$.

In soliton analysis the most common form of NLS equation is the following:

$$i\frac{\partial U\left(Z,T\right)}{\partial Z}+\frac{\partial^2 U\left(Z,T\right)}{\partial T^2}+\left|U\left(Z,T\right)\right|^2 U\left(Z,T\right)=0 \qquad \text{[IX-36]}$$

where $U\left(Z,T\right)=N\;u\left(Z,T\right)\qquad \vec{\mathbf{G}}\left(\varsigma,\tau\right)\sqrt{\dfrac{\gamma\,\Delta\tau^2}{|\beta_2|}}$.

If $N\geq 1$ the following ***fundamental soliton*** will propagate undistorted for an arbitrary distance:

$$U\left(Z,T\right)=\mathrm{sec}h\left(T\right)\,\exp\left(i\;Z/2\right) \qquad \text{[IX-37]}$$

If $N\geq 2$ the following ***second-order soliton*** will propagate undistorted for an arbitrary distance:

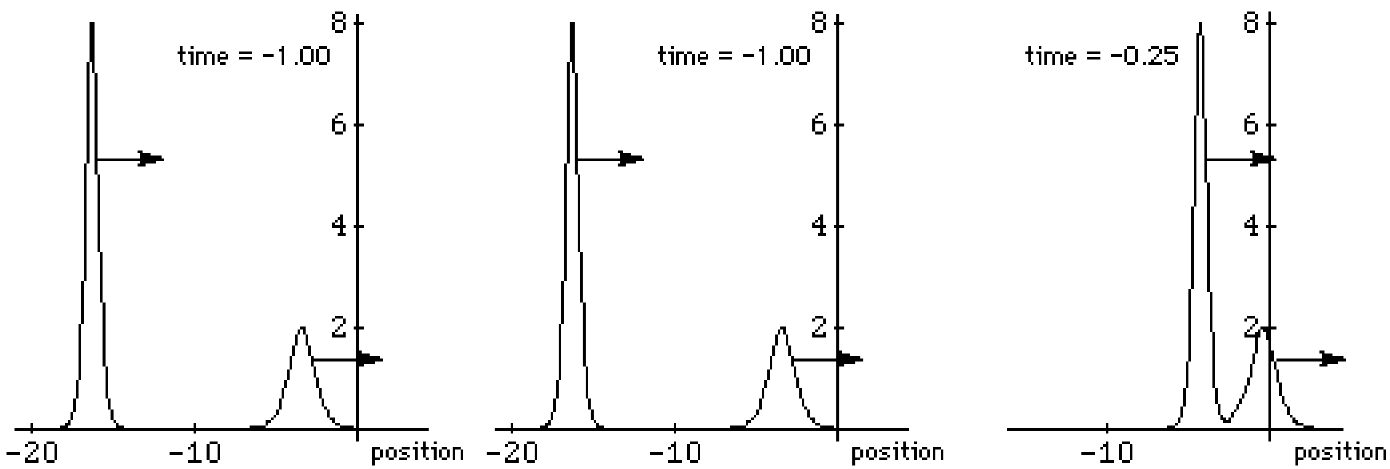
$$U\left(Z,T\right)=\frac{4\left[\cosh\left(3\,T\right)+3\exp\left(i\;4\,Z\right)\,\cosh\left(T\right)\right]}{\left[\cosh\left(4\,T\right)+4\,\cosh\left(2\,T\right)+3\,\cos\left(4\,Z\right)\right]}\,\exp\left(i\;Z/2\right) \qquad \text{[IX-37]}$$

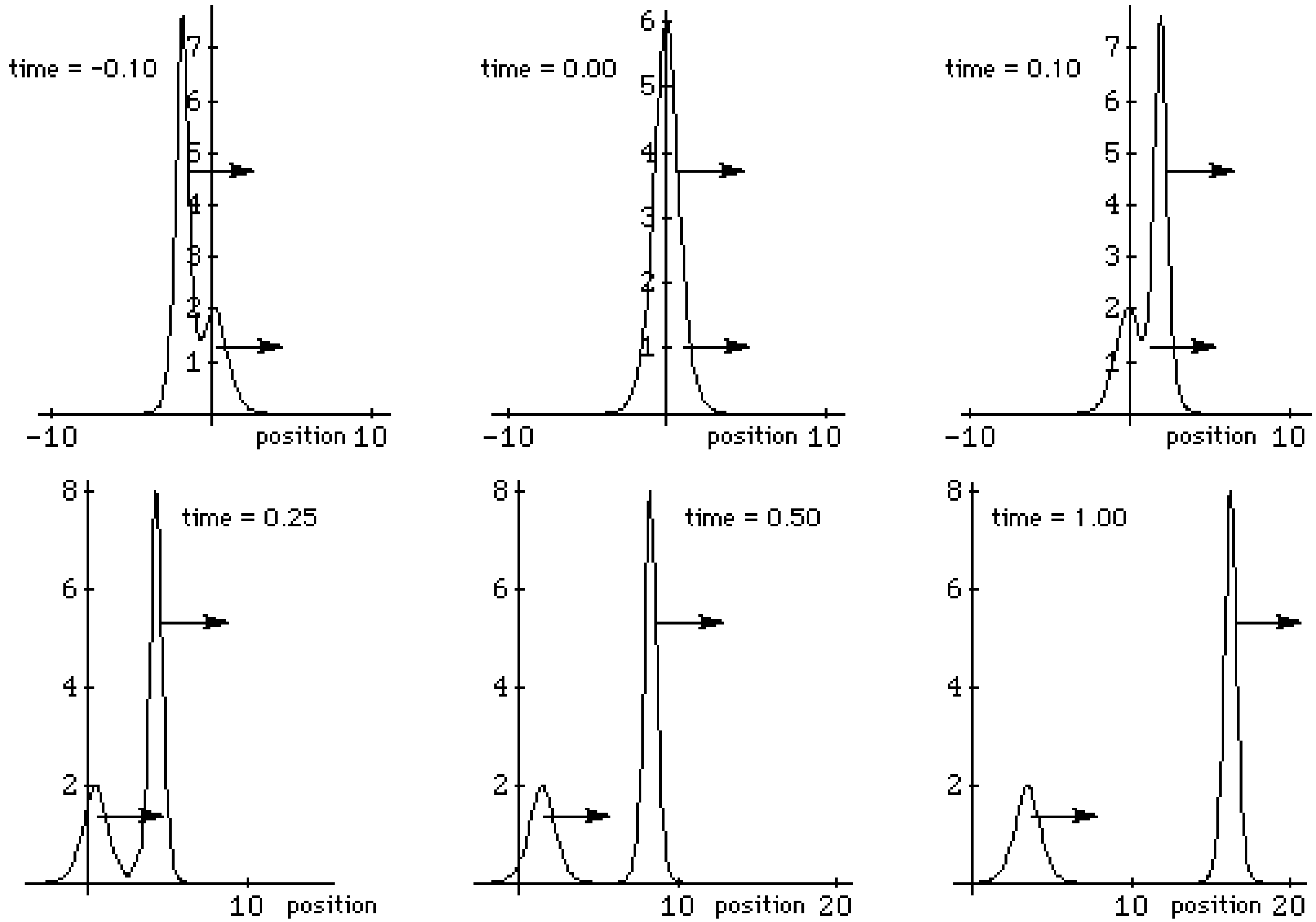
$$\left|U\left(Z,T\right)\right|^2=\frac{16\left[\cosh^2\left(3\,T\right)+9\cosh^2\left(T\right)+6\,\cos\left(4\;Z\right)\,\cosh\left(3\,T\right)\,\cosh\left(T\right)\right]}{\left[\cosh\left(4\,T\right)+4\,\cosh\left(2\,T\right)+3\cos\left(4\,Z\right)\right]^2} \qquad \text{[IX-37]}$$

TWO SOLITON COLLISION ILLUSTRATIONS

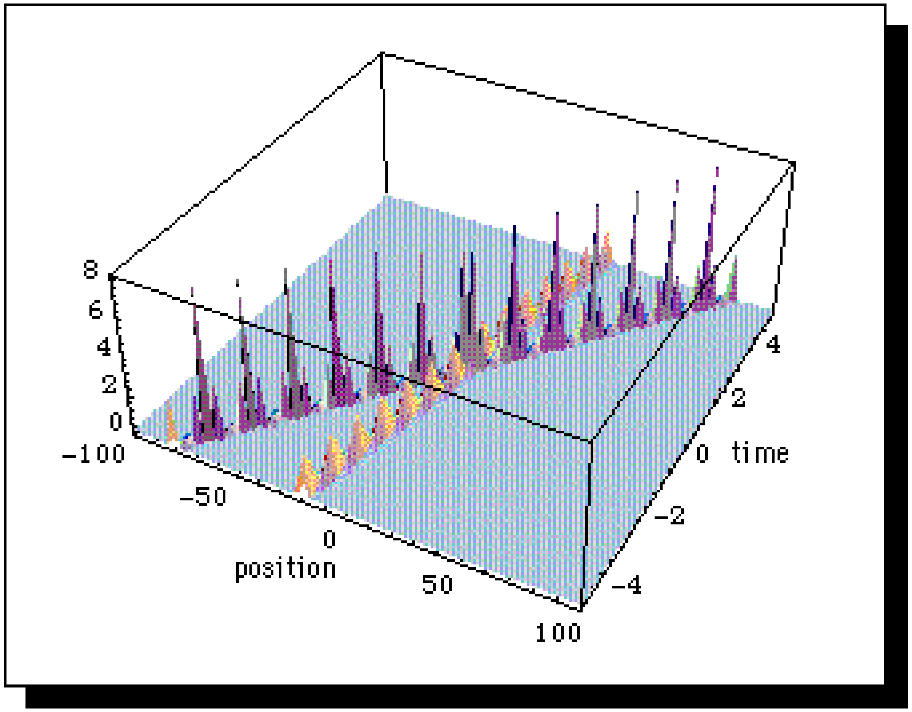
The following are time-lapsed illustration of the propagation and collision of two solitons -- plot here is

$$u(x,t) = 12\,\frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\left[3\cosh(x - 28t) + \cosh(3x - 36t)\right]^2}$$





time = -0.10
time = 0.00
time = 0.10
time = 0.25
time = 0.50
time = 1.00
position
-10
10
20



8
6
4
2
0
-100
-50
0
position
50
100
4
2
0
time
-2
-4