# VIII. GUIDED WAVES IN PLANAR STRUCTURES

## CHARACTERISTICS OF PLANE WAVE SOLUTIONS:

For the record, let us restate the frequency domain, macroscopic Maxwell's equations which are valid in the high frequency or *optical regime* for a linear, local, isotropic medium -- *viz*.

$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=\vec{\nabla}\times\varepsilon^{-1}(\vec{\mathbf{r}},\omega)\;\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)=-i\,\omega\,\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega)=-i\,\omega\mu_0\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[ VIII- 1a ]}$$

$$\begin{aligned}\vec{\nabla}\times\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega)&=\mu_0\,\vec{\nabla}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=\mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)+i\,\omega\mu_0\varepsilon(\vec{\mathbf{r}},\omega)\;\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)\\&=\mu_0\,\vec{\mathbf{J}}(\vec{\mathbf{r}},\omega)+i\omega\mu_0\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)\end{aligned} \qquad \text{[ VIII- 1b ]}$$

$$\vec{\nabla}\cdot\;\varepsilon(\vec{\mathbf{r}},\omega)\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=\vec{\nabla}\cdot\;\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)=\rho(\vec{\mathbf{r}},\omega) \qquad \text{[ VIII- 1c ]}$$

$$\vec{\nabla}\cdot\;\vec{\mathbf{B}}(\vec{\mathbf{r}},\omega)=\mu_0\,\vec{\nabla}\cdot\;\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[ VIII- 1d ]}$$

Further, in regions free of **explicit sources** of current and charge we may write

$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=-i\,\omega\mu_0\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega) \qquad \text{[ VIII- 2a ]}$$

$$\vec{\nabla}\times\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=i\,\omega\varepsilon_{\mathrm{eff}}(\vec{\mathbf{r}},\omega)\;\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[ VIII- 2b ]}$$

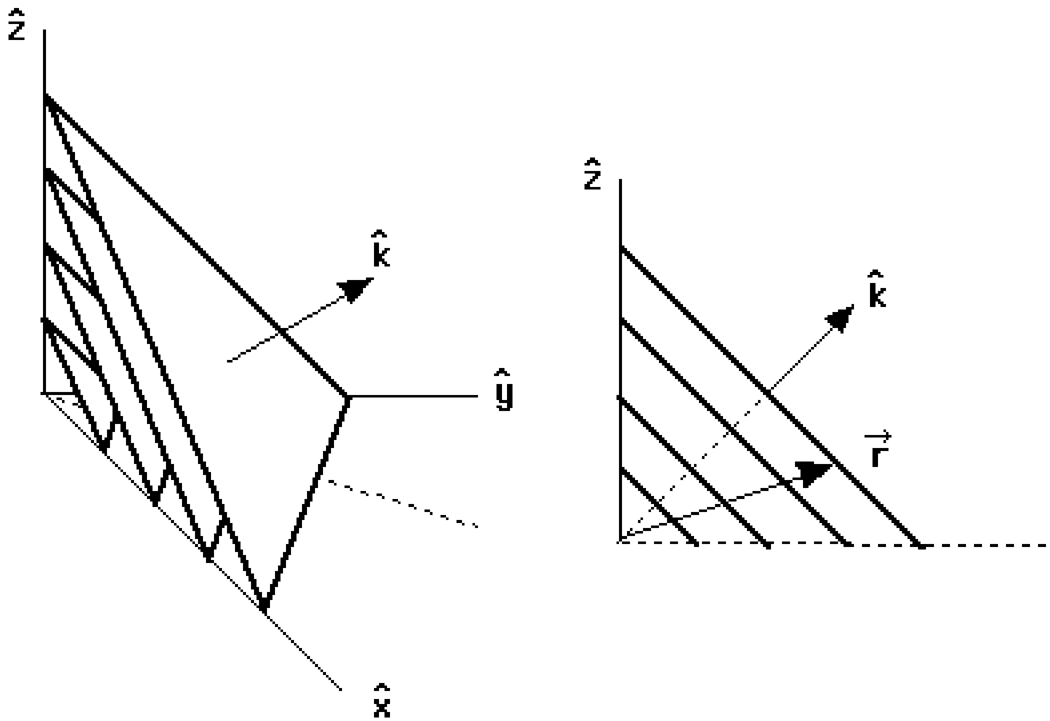
$$\vec{\nabla}\cdot\;\varepsilon(\vec{\mathbf{r}},\omega)\;\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[ VIII- 2c ]}$$

$$\vec{\nabla}\cdot\;\vec{\mathbf{H}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[ VIII- 2d ]}$$

where $\varepsilon_{\mathrm{eff}}(\vec{\mathbf{r}},\omega)\equiv\varepsilon_{\mathrm{eff}}(\vec{\mathbf{r}},\omega)-i\,\sigma(\vec{\mathbf{r}},\omega)/\omega$. In this set of lectures it is our intention to **explore in some depth** plane wave propagation within a uniform medium -- *i.e.* $\varepsilon_{\mathrm{eff}}(\vec{\mathbf{r}},\omega)\equiv\varepsilon_{\mathrm{eff}}(\omega)$. To that end we consider a plane wave solution in the form

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\mathbf{E}}(\omega)\ \exp\left(-i\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right) = \vec{\mathbf{E}}(\omega)\ \exp\left[-i\left(x\,k_x + y\ k_y + z\ k_z\right)\right] \qquad \text{[ VIII- 3 ]}$$

which may pictorially represented as




Therefore

$$\vec{\nabla}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\nabla}\cdot\left[\vec{\mathbf{E}}(\omega)\ \exp\left(-i\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right)\right] = -i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[ VIII- 4a ]}$$

$$\vec{\nabla}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) = \vec{\nabla}\times\left[\vec{\mathbf{E}}(\omega)\ \exp\left(-i\,\vec{\mathbf{r}}\cdot\vec{\mathbf{k}}\right)\right] = -i\,\vec{\mathbf{k}}\times\vec{\mathbf{E}}(\vec{\mathbf{r}},\omega) \qquad \text{[ VIII- 4b ]}$$

and the Maxwell's equations formulated in Equation [VIII-2 ] become

$$-i\,\vec{\mathbf{k}}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)=-i\,\omega\mu_0\vec{\mathbf{H}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[ VIII- 5a ]}$$

$$-i\,\vec{\mathbf{k}}\times\vec{\mathbf{H}}\left(\vec{\mathbf{r}},\omega\right)=i\,\omega\varepsilon\left(\omega\right)\,\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[ VIII- 5b ]}$$

$$-i\,\vec{\mathbf{k}}\ \ \vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)=0 \qquad \text{[ VIII- 5c ]}$$

$$-i\,\vec{\mathbf{k}}\ \ \vec{\mathbf{H}}\left(\vec{\mathbf{r}},\omega\right)=0 \qquad \text{[ VIII- 5d ]}$$

Operate through on both sides of Equation [ VIII- 5a ] with the operator "$\vec{\mathbf{k}}\times$" we obtain

$$\vec{\mathbf{k}}\times\left[\vec{\mathbf{k}}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)\right]=\omega\mu_0\,\vec{\mathbf{k}}\times\vec{\mathbf{H}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[ VIII- 6a ]}$$

Using the "bac-cab" rule[30] and Equation [ VIII- 5b ] this becomes

$$\vec{\mathbf{k}}\left[\vec{\mathbf{k}}\ \ \vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)\right]-\left[\vec{\mathbf{k}}\ \ \vec{\mathbf{k}}\right]\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)=-\omega^2\mu_0\,\varepsilon_{\mathrm{eff}}\left(\omega\right)\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[ VIII- 6b ]}$$

or finally

$$\left[\vec{\mathbf{k}}\ \ \vec{\mathbf{k}}\right]\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)=\omega^2\mu_0\,\varepsilon_{\mathrm{eff}}\left(\omega\right)\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right) \qquad k^2=\omega^2\mu_0\,\varepsilon_{\mathrm{eff}}\left(\omega\right) \qquad \text{[ VIII- 6c ]}$$

Substituting these results into Equation [ VIII- 5a ] we obtain

$$\vec{\mathbf{H}}\left(\vec{\mathbf{r}},\omega\right)=\left(\omega\mu_0\right)^{-1}k\left[\hat{\mathbf{k}}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)\right]=\sqrt{\varepsilon_{\mathrm{eff}}\left(\omega\right)/\mu_0}\left[\hat{\mathbf{k}}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)\right] \qquad \text{[ VIII- 7 ]}$$

so that the **wave impedance** is given by

$$\eta\left(\omega\right)=\left|\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)\right|/\left|\vec{\mathbf{H}}\left(\vec{\mathbf{r}},\omega\right)\right|=\sqrt{\mu_0/\varepsilon_{\mathrm{eff}}\left(\omega\right)}\ . \qquad \text{[ VIII- 11 ]}$$

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30 That is $\vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right)=\vec{\mathbf{b}}\left(\vec{\mathbf{a}}\ \ \vec{\mathbf{c}}\right)-\vec{\mathbf{c}}\left(\vec{\mathbf{a}}\ \ \vec{\mathbf{b}}\right)$.

Thus, the complete expression for an electromagnetic plane wave propagating in a direction $\hat{\mathbf{k}}$ in a uniform medium is given by

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = \vec{\mathbf{E}}(\omega)\exp\left[-j\left(\vec{\mathbf{r}}\;\;\vec{\mathbf{k}} - \omega\, t\right)\right] \qquad \text{[ VIII- 9a ]}$$

$$\vec{\mathbf{H}}(\vec{\mathbf{r}}, t) = \left[\eta(\omega)\right]^{-1}\left[\hat{\mathbf{k}} \times \vec{\mathbf{E}}(\vec{\mathbf{r}}, \omega)\right] \qquad \text{[ VIII- 9b ]}$$

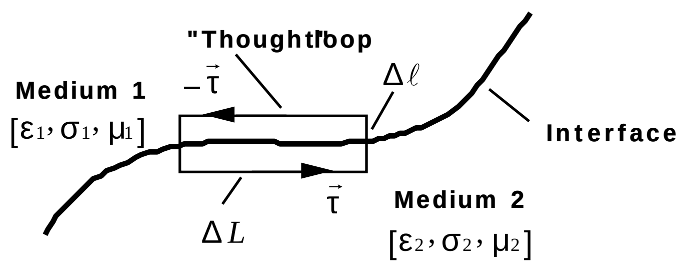
**<u>ELECTROMAGNETIC INTERFACIAL CONTINUITY CONDITIONS:</u>**

The previous section gives a **complete** plane wave solution within a **particular** uniform, linear, isotropic medium. The key remaining problem is to find how that solution may be extended into a second uniform, linear, isotropic medium. The conditions for extending the solution across an interface between two materials are give by consideration of the appropriate integral forms of Maxwell's equations -- *viz*.

$$\oint \vec{\mathbf{E}}(\vec{\mathbf{r}}, t)\;\; d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{\mathbf{B}}(\vec{\mathbf{r}}, t)\;\; d\vec{A} \qquad \text{[ VIII- 10a ]}$$

$$\oint \vec{\mathbf{H}}(\vec{\mathbf{r}}, t)\;\; d\vec{l} = \iint \vec{\mathbf{J}}(\vec{\mathbf{r}}, t)\;\; d\vec{A} + \frac{\partial}{\partial t}\iint \vec{\mathbf{D}}(\vec{\mathbf{r}}, t)\;\; d\vec{A} \qquad \text{[ VIII- 10b ]}$$

Applying these equations to the small ***thought* loop** that spans the interfacial surface, as illustrated below




it is seen that Equation [ VIII- 10a ] yields

$$\oint \vec{\mathbf{E}}(\vec{\mathbf{r}}, t) \quad d\vec{l} \qquad \left\{ \vec{\mathbf{E}}_2(\vec{\mathbf{r}}, t) - \vec{\mathbf{E}}_1(\vec{\mathbf{r}}, t) \right\} \quad \vec{\tau}\ \Delta L \qquad 0 \qquad \text{[ VIII- 11 ]}$$

unless $\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$ is **pathologically** large over the loop. Similarly, it is seen that Equation [ VIII- 10b ] yields

$$\oint \vec{\mathbf{H}}(\vec{\mathbf{r}}, t) \quad d\vec{l} \qquad \left\{ \vec{\mathbf{H}}_2(\vec{\mathbf{r}}, t) - \vec{\mathbf{H}}_1(\vec{\mathbf{r}}, t) \right\} \quad \vec{\tau}\ \Delta L \qquad 0 \qquad \text{[ VIII- 12 ]}$$

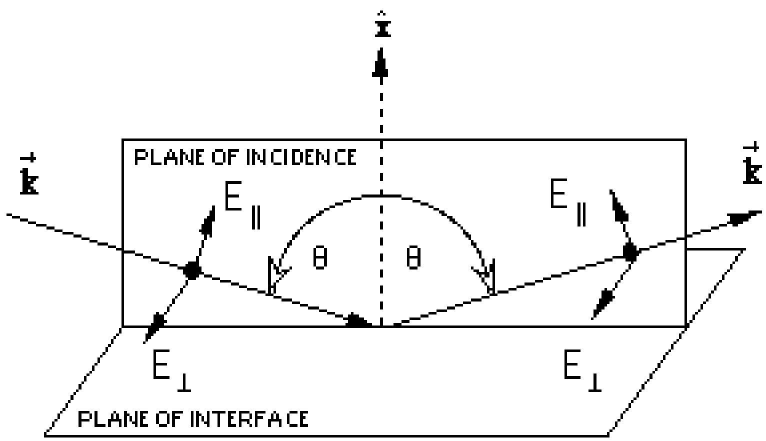
unless $\vec{\mathbf{J}}(\vec{\mathbf{r}}, t)$ and/or $\vec{\mathbf{D}}(\vec{\mathbf{r}}, t)$ are **pathologically** large over the loop.

> In words and in general, **the tangential component of the electric field strength $\vec{\mathbf{E}}(\vec{\mathbf{r}}, t)$ and the magnetic field strength $\vec{\mathbf{H}}(\vec{\mathbf{r}}, t)$ are continuous across an interfacial surface between two materials unless the electric current density $\vec{\mathbf{J}}(\vec{\mathbf{r}}, t)$, the magnetic flux density $\vec{\mathbf{B}}(\vec{\mathbf{r}}, t)$, or the electric flux density $\vec{\mathbf{D}}(\vec{\mathbf{r}}, t)$ are pathologically large near that interfacial surface.**

## THE FRESNEL EQUATIONS:

Consider then a plane wave incident on a planar interfacial surface.

### The Spatial Configuration:[31]




## The Mathematical Representation of Fields:

In abstract vector form, the incident field is given by[32]

$$\vec{\mathbf{E}}^{\,inc} = \left\{\vec{\mathbf{E}}^{\,inc} - \eta_1\, \hat{\mathbf{k}}^{inc} \times \vec{\mathbf{H}}^{inc}_{||}\right\} \exp\left(-i\, k_1\, \hat{\mathbf{k}}^{inc}\ \vec{\mathbf{r}}\right)$$
$$\vec{\mathbf{H}}^{inc} = \left\{\vec{\mathbf{H}}^{inc}_{||} + \eta_1^{-1}\, \hat{\mathbf{k}}^{inc} \times \vec{\mathbf{E}}^{\,inc}\right\} \exp\left(-i\, k_1\, \hat{\mathbf{k}}^{inc}\ \vec{\mathbf{r}}\right) \qquad [\text{ VIII- 13a }]$$

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31 Note: In this figure we have taken the *plane of reflection* to be identical to the *plane of incidence*. While assumed here for simplicity, this important identity is establish in the analysis below.

32 A note on notation: The subscripts and || refer to the polariztion of the electric field taken with respect to the *plane of incidence*. The field components are also called *transverse electric* or TE components and the || field components are called *transverse magnetic* or TM components.

the reflected field is given by

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{ref}} &= \left\{\vec{\mathbf{E}}^{\,\mathrm{ref}} - \eta_1\,\hat{\mathbf{k}}^{\mathrm{ref}} \times \vec{\mathbf{H}}_{||}^{\mathrm{ref}}\right\} \exp\left(-i\,k_1\,\hat{\mathbf{k}}^{\mathrm{ref}}\,\vec{\mathbf{r}}\right)\\ \vec{\mathbf{H}}^{\mathrm{ref}} &= \left\{\vec{\mathbf{H}}_{||}^{\mathrm{ref}} + \eta_1^{-1}\,\hat{\mathbf{k}}^{\mathrm{ref}} \times \vec{\mathbf{E}}^{\,\mathrm{ref}}\right\} \exp\left(-i\,k_1\,\hat{\mathbf{k}}^{\mathrm{ref}}\,\vec{\mathbf{r}}\right)\end{aligned} \qquad \text{[ VIII- 14a ]}$$

and the transmitted field is given by

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{tran}} &= \left\{\vec{\mathbf{E}}^{\,\mathrm{tran}} - \eta_2\,\hat{\mathbf{k}}^{\mathrm{tran}} \times \vec{\mathbf{H}}_{||}^{\mathrm{tran}}\right\} \exp\left(-i\,k_2\,\hat{\mathbf{k}}^{\mathrm{tran}}\,\vec{\mathbf{r}}\right)\\ \vec{\mathbf{H}}^{\mathrm{tran}} &= \left\{\vec{\mathbf{H}}_{||}^{\mathrm{tran}} + \eta_2^{-1}\,\hat{\mathbf{k}}^{\mathrm{tran}} \times \vec{\mathbf{E}}^{\,\mathrm{tran}}\right\} \exp\left(-i\,k_2\,\hat{\mathbf{k}}^{\mathrm{tran}}\,\vec{\mathbf{r}}\right)\end{aligned} \qquad \text{[ VIII- 15a ]}$$

In coordinate form these equations become:

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{inc}} &= \left\{\mathrm{E}^{\mathrm{inc}}\hat{\mathbf{y}} - \eta_1\left[-\cos\theta_{\mathrm{inc}}\;\hat{\mathbf{x}} + \sin\theta_{\mathrm{inc}}\;\hat{\mathbf{z}}\right]\times\left[\mathrm{H}_{||}^{\mathrm{inc}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_1\left(-x\cos\theta_{\mathrm{inc}} + z\sin\theta_{\mathrm{inc}}\right)\right]\\ \vec{\mathbf{H}}^{\mathrm{inc}} &= \left\{\mathrm{H}_{||}^{\mathrm{inc}}\hat{\mathbf{y}} + \eta_1^{-1}\left[-\cos\theta_{\mathrm{inc}}\;\hat{\mathbf{x}} + \sin\theta_{\mathrm{inc}}\;\hat{\mathbf{z}}\right]\times\left[\mathrm{E}^{\mathrm{inc}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_1\left(-x\cos\theta_{\mathrm{inc}} + z\sin\theta_{\mathrm{inc}}\right)\right]\end{aligned} \qquad \text{[ VIII- 13b ]}$$

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{ref}} &= \left\{\mathrm{E}^{\mathrm{ref}}\hat{\mathbf{y}} - \eta_1\left[\cos\theta_{\mathrm{ref}}\;\hat{\mathbf{x}} + \sin\theta_{\mathrm{ref}}\;\hat{\mathbf{z}}\right]\times\left[\mathrm{H}_{||}^{\mathrm{ref}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_1\left(x\cos\theta_{\mathrm{ref}} + z\sin\theta_{\mathrm{ref}}\right)\right]\\ \vec{\mathbf{H}}^{\mathrm{ref}} &= \left\{\mathrm{H}_{||}^{\mathrm{ref}}\hat{\mathbf{y}} + \eta_1^{-1}\left[\cos\theta_{\mathrm{ref}}\;\hat{\mathbf{x}} + \sin\theta_{\mathrm{ref}}\;\hat{\mathbf{z}}\right]\times\left[\mathrm{E}^{\mathrm{ref}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_1\left(x\cos\theta_{\mathrm{ref}} + z\sin\theta_{\mathrm{ref}}\right)\right]\end{aligned} \qquad \text{[ VIII- 14b ]}$$

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{tran}} &= \left\{\mathrm{E}^{\mathrm{tran}}\hat{\mathbf{y}} - \eta_2\left[-\cos\theta_{\mathrm{tran}}\;\hat{\mathbf{x}} + \sin\theta_{\mathrm{tran}}\;\hat{\mathbf{z}}\right]\times\left[\mathrm{H}_{||}^{\mathrm{tran}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_2\left(-x\cos\theta_{\mathrm{tran}} + z\sin\theta_{\mathrm{tran}}\right)\right]\\ \vec{\mathbf{H}}^{\mathrm{tran}} &= \left\{\mathrm{H}_{||}^{\mathrm{tran}}\hat{\mathbf{y}} + \eta_2^{-1}\left[-\cos\theta_{\mathrm{tran}}\;\hat{\mathbf{x}} + \sin\theta_{\mathrm{tran}}\;\hat{\mathbf{z}}\right]\times\left[\mathrm{E}^{\mathrm{tran}}\hat{\mathbf{y}}\right]\right\} \exp\left[-i\,k_2\left(-x\cos\theta_{\mathrm{tran}} + z\sin\theta_{\mathrm{tran}}\right)\right]\end{aligned} \qquad \text{[ VIII- 15b ]}$$

Or expanding out the cross-products:

$$\begin{aligned}\vec{\mathbf{E}}^{\,\mathrm{inc}} &= \left\{\mathrm{E}^{\mathrm{inc}}\hat{\mathbf{y}} + \left(\eta_1\,\mathrm{H}_{||}^{\mathrm{inc}}\right)\left[\cos\theta_{\mathrm{inc}}\;\hat{\mathbf{z}} + \sin\theta_{\mathrm{inc}}\;\hat{\mathbf{x}}\right]\right\} \exp\left[-i\,k_1\left(-x\cos\theta_{\mathrm{inc}} + z\sin\theta_{\mathrm{inc}}\right)\right]\\ \vec{\mathbf{H}}^{\mathrm{inc}} &= \left\{\mathrm{H}_{||}^{\mathrm{inc}}\hat{\mathbf{y}} - \left(\eta_1^{-1}\,\mathrm{E}^{\mathrm{inc}}\right)\left[\cos\theta_{\mathrm{inc}}\;\hat{\mathbf{z}} + \sin\theta_{\mathrm{inc}}\;\hat{\mathbf{x}}\right]\right\} \exp\left[-i\,k_1\left(-x\cos\theta_{\mathrm{inc}} + z\sin\theta_{\mathrm{inc}}\right)\right]\end{aligned} \qquad \text{[ VIII- 13c ]}$$

$$\vec{\mathbf{E}}^{\,\mathrm{ref}} = \left\{ \mathrm{E}^{\mathrm{ref}} \hat{\mathbf{y}} + \left( \eta_1 \, \mathrm{H}_{||}^{\mathrm{ref}} \right) \left[ -\cos\theta_{\mathrm{ref}} \, \hat{\mathbf{z}} + \sin\theta_{\mathrm{ref}} \, \hat{\mathbf{x}} \right] \right\} \exp\left[ -i \, k_1 \left( x \cos\theta_{\mathrm{ref}} + z \sin\theta_{\mathrm{ref}} \right) \right]$$

$$\vec{\mathbf{H}}^{\,\mathrm{ref}} = \left\{ \mathrm{H}_{||}^{\mathrm{ref}} \hat{\mathbf{y}} - \left( \eta_1^{-1} \, \mathrm{E}^{\mathrm{ref}} \right) \left[ -\cos\theta_{\mathrm{ref}} \, \hat{\mathbf{z}} + \sin\theta_{\mathrm{ref}} \, \hat{\mathbf{x}} \right] \right\} \exp\left[ -i \, k_1 \left( x \cos\theta_{\mathrm{ref}} + z \sin\theta_{\mathrm{ref}} \right) \right] \qquad \text{[ VIII- 14c ]}$$

$$\vec{\mathbf{E}}^{\,\mathrm{tran}} = \left\{ \mathrm{E}^{\mathrm{tran}} \hat{\mathbf{y}} + \left( \eta_2 \, \mathrm{H}_{||}^{\mathrm{tran}} \right) \left[ \cos\theta_{\mathrm{tran}} \, \hat{\mathbf{z}} + \sin\theta_{\mathrm{tran}} \, \hat{\mathbf{x}} \right] \right\} \exp\left[ -i \, k_2 \left( -x \cos\theta_{\mathrm{tran}} + z \sin\theta_{\mathrm{tran}} \right) \right]$$

$$\vec{\mathbf{H}}^{\,\mathrm{tran}} = \left\{ \mathrm{H}_{||}^{\mathrm{tran}} \hat{\mathbf{y}} - \left( \eta_2^{-1} \, \mathrm{E}^{\mathrm{tran}} \right) \left[ \cos\theta_{\mathrm{tran}} \, \hat{\mathbf{z}} + \sin\theta_{\mathrm{tran}} \, \hat{\mathbf{x}} \right] \right\} \exp\left[ -i \, k_2 \left( -x \cos\theta_{\mathrm{tran}} + z \sin\theta_{\mathrm{tran}} \right) \right] \qquad \text{[ VIII- 15c ]}$$

Applying any kind of continuity conditions at the interface requires that

$$\theta_{\mathrm{ref}} = \theta_{\mathrm{inc}} \qquad \textit{Law of Sinus} \qquad \text{[ VIII- 16a ]}$$

$$k_2 \sin\theta_{\mathrm{tran}} = k_1 \sin\theta_{\mathrm{inc}} \qquad \textit{Law of Snell} \qquad \text{[ VIII- 16b ]}$$

Applying, in particular, the continuity conditions discussed in the previous section -- *viz*.

$$\left[ \vec{\mathbf{E}}^{1} \right]_{\mathrm{tang}} = \left[ \vec{\mathbf{E}}^{2} \right]_{\mathrm{tang}} \quad \text{and} \quad \left[ \vec{\mathbf{H}}^{1} \right]_{\mathrm{tang}} = \left[ \vec{\mathbf{H}}^{2} \right]_{\mathrm{tang}} \qquad \text{[ VIII- 17 ]}$$

at the interface, requires that

$$\begin{gathered} \mathrm{E}^{\mathrm{inc}} + \mathrm{E}^{\mathrm{ref}} = \mathrm{E}^{\mathrm{tran}} \\ \eta_1^{-1} \cos\theta_{\mathrm{inc}} \left[ \mathrm{E}^{\mathrm{inc}} - \mathrm{E}^{\mathrm{ref}} \right] = \eta_2^{-1} \cos\theta_{\mathrm{tran}} \left[ \mathrm{E}^{\mathrm{tran}} \right] \end{gathered} \qquad \text{[ VIII- 18 ]}$$

and that

$$\begin{gathered} \mathrm{H}_{||}^{\mathrm{inc}} + \mathrm{H}_{||}^{\mathrm{ref}} = \mathrm{H}_{||}^{\mathrm{tran}} \\ \eta_1 \cos\theta_{\mathrm{inc}} \left[ \mathrm{H}_{||}^{\mathrm{inc}} - \mathrm{H}_{||}^{\mathrm{ref}} \right] = \eta_2 \cos\theta_{\mathrm{tran}} \left[ \mathrm{H}_{||}^{\mathrm{tran}} \right] \end{gathered} \qquad \text{[ VIII- 19 ]}$$

These two sets of equations yield the **Fresnel Reflection Equations** -- *viz*.

$$\frac{\mathrm{E}^{\mathrm{ref}}}{\mathrm{E}^{\mathrm{inc}}} = \left[ \frac{\eta_1^{-1} \cos\theta_{\mathrm{inc}} - \eta_2^{-1} \cos\theta_{\mathrm{tran}}}{\eta_1^{-1} \cos\theta_{\mathrm{inc}} + \eta_2^{-1} \cos\theta_{\mathrm{tran}}} \right] \qquad \text{[ VIII- 20a ]}$$

and

$$\frac{H_{||}^{ref}}{H_{||}^{inc}} = \left[\frac{\eta_1 \cos\theta_{inc} - \eta_2 \cos\theta_{tran}}{\eta_1 \cos\theta_{inc} + \eta_2 \cos\theta_{tran}}\right] \qquad \text{[ VIII- 21a ]}$$

Since $\eta_1^{-1} \sin\theta_{inc} = \eta_2^{-1} \sin\theta_{tran}$

$$\frac{E^{ref}}{E^{inc}} = \frac{\cos\theta_{inc} \sin\theta_{tran} - \cos\theta_{tran} \sin\theta_{inc}}{\cos\theta_{inc} \sin\theta_{tran} + \cos\theta_{tran} \sin\theta_{inc}} = \frac{\sin\left(\theta_{tran} - \theta_{inc}\right)}{\sin\left(\theta_{tran} + \theta_{inc}\right)} \qquad \text{[ VIII- 20b ]}$$

and

$$\frac{H_{||}^{ref}}{H_{||}^{inc}} = \frac{\cos\theta_{inc} \sin\theta_{inc} - \cos\theta_{tran} \sin\theta_{tran}}{\cos\theta_{inc} \sin\theta_{inc} + \cos\theta_{tran} \sin\theta_{tran}} = \frac{\tan\left(\theta_{inc} - \theta_{tran}\right)}{\tan\left(\theta_{inc} + \theta_{tran}\right)} \qquad \text{[ VIII- 21b ]}$$

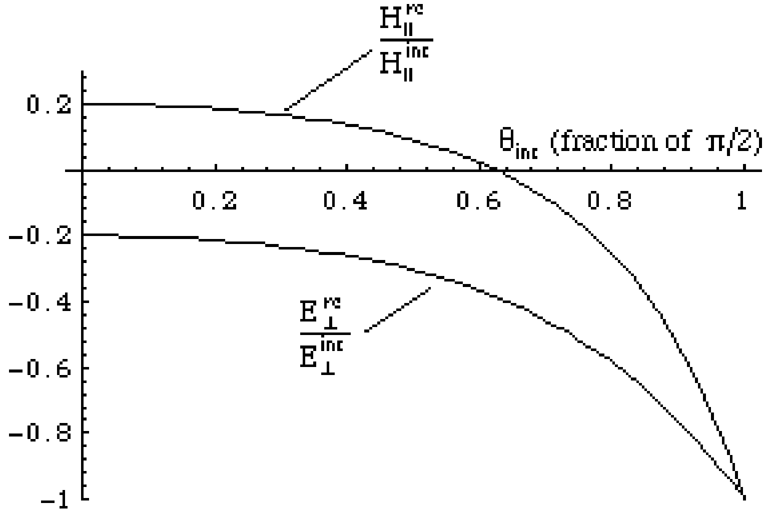
These equations taken together with first equations from Equations [ VIII- 18 ] and [ VIII- 19 ] yield the <u>**Fresnel Transmission Equations**</u> -- *viz*

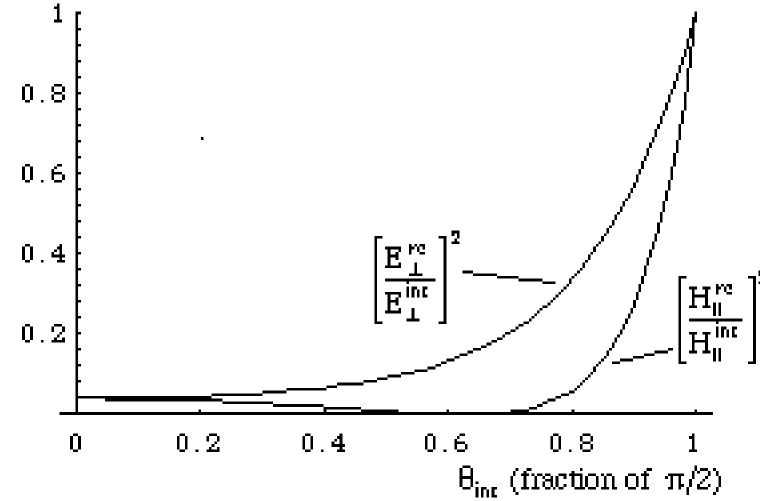
$$\frac{E^{tran}}{E^{inc}} = \frac{2\cos\theta_{inc} \sin\theta_{tran}}{\cos\theta_{inc} \sin\theta_{tran} + \cos\theta_{tran} \sin\theta_{inc}} \qquad \text{[ VIII- 22 ]}$$

and

$$\frac{H_{||}^{tran}}{H_{||}^{inc}} = \frac{2\cos\theta_{inc} \sin\theta_{inc}}{\cos\theta_{inc} \sin\theta_{inc} + \cos\theta_{tran} \sin\theta_{tran}} \qquad \text{[ VIII- 23 ]}$$

### ***FAMOUS FRESNEL REFLECTION CURVES*** ( $n_2/n_1 = \eta_1/n_2 = \sqrt{\varepsilon_2/\varepsilon_1} = 1.5$ )




The minimum (zero) in $H_{||}^{ref}/H_{||}^{inc}$ occurs at the Brewster angle where

$$\tan\left(\theta_{inc}^{Brewster}+\theta_{tran}^{Brewster}\right) \quad \infty \qquad \text{[ VIII- 24a ]}$$

or

$$\theta_{tran}^{Brewster} = \pi/2 - \theta_{inc}^{Brewste} \qquad \text{[ VIII- 24b ]}$$

or (from Snell's equation)

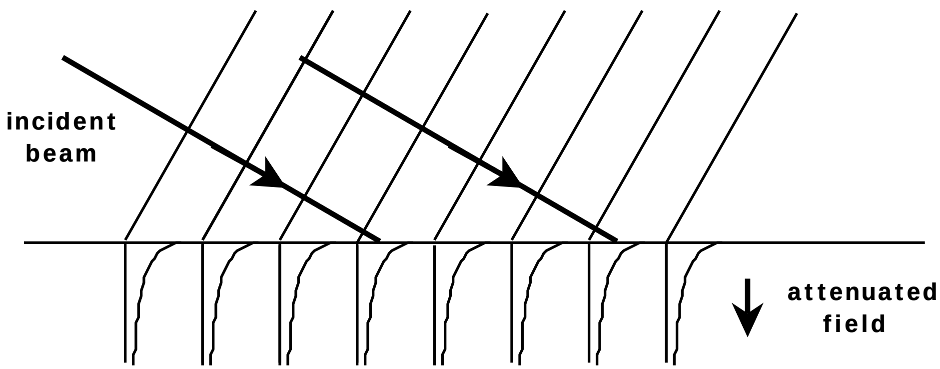
$$\tan\theta_{inc}^{Brewster} = \eta_1/\eta_2 = n_2/n_1 = \sqrt{\varepsilon_2/\varepsilon_1} \; . \qquad \text{[ VIII- 24c ]}$$

## Total Internal Reflection

Reconsider Equation [ VIII- 15c ] and use Snell's law to write the exponential factors in the form

$$\vec{\mathbf{E}}^{\,\mathrm{tran}} = \left\{ \mathrm{E}^{\mathrm{tran}}\hat{\mathbf{y}} + \left(\eta_2\, \mathrm{H}_{||}^{\mathrm{tran}}\right)\left[\cos\theta_{\mathrm{tran}}\,\hat{\mathbf{z}} + \sin\theta_{\mathrm{tran}}\,\hat{\mathbf{x}}\right]\right\} \exp\left[i x \sqrt{k_2^2 - k_1^2 \sin^2\theta_{\mathrm{inc}}}\right] \exp\left[-i z\; k_1 \sin\theta_{\mathrm{inc}}\right] \qquad \text{[ VIII- 25 ]}$$

When $\sin\theta_{\mathrm{inc}} > k_2/k_1 = n_2/n_1 \equiv \sin\theta_{\mathrm{inc}}^{\mathrm{crit}}$, $\vec{\mathbf{E}}^{\,\mathrm{tran}}$, the solution in medium 2, is **attenuated**!




Reconsideration of Equation [ VIII- 20a ] and [ VIII- 21a ] shows that the magnitude of the reflection coefficients are **one** when $\sin\theta_{\mathrm{inc}} > \sin\theta_{\mathrm{inc}}^{\mathrm{crit}}$ -- *viz.*

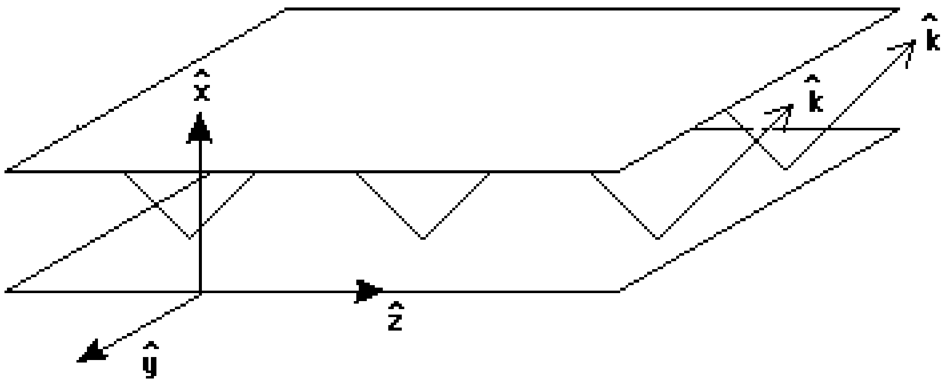
$$\frac{\mathrm{E}^{\mathrm{ref}}}{\mathrm{E}^{\mathrm{inc}}} = \left[\frac{\cos\theta_{\mathrm{inc}} - i\sqrt{\sin^2\theta_{\mathrm{inc}} - \left(k_2/k_1\right)^2}}{\cos\theta_{\mathrm{inc}} + i\sqrt{\sin^2\theta_{\mathrm{inc}} - \left(k_2/k_1\right)^2}}\right] = \exp\left\{-i\, 2\tan^{-1}\left[\frac{\sqrt{\sin^2\theta_{\mathrm{inc}} - \left(k_2/k_1\right)^2}}{\cos\theta_{\mathrm{inc}}}\right]\right\} \qquad \text{[ VIII- 26a ]}$$

and

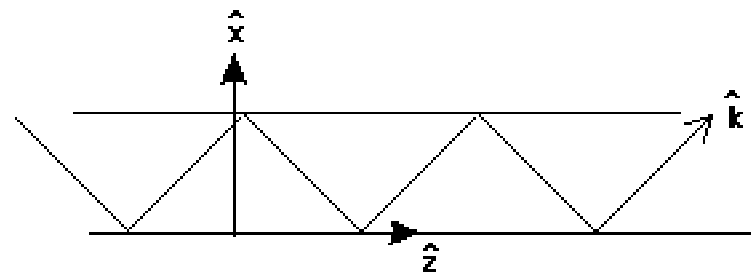
$$\frac{H_{\parallel}^{ref}}{H_{\parallel}^{inc}} = \left[\frac{\cos\theta_{inc} - j\sqrt{\left(k_1/k_2\right)^2 \sin^2\theta_{inc} - 1}}{\cos\theta_{inc} + j\sqrt{\left(k_1/k_2\right)^2 \sin^2\theta_{inc} - 1}}\right] = \exp\left\{ -j\, 2 \tan^{-1}\left[\frac{\sqrt{\left(k_1/k_2\right)^2 \sin^2\theta_{inc} - 1}}{\cos\theta_{inc}}\right]\right\}. \quad [\text{ VIII- 26b }]$$

## PARALLEL PLATE WAVEGUIDE:

Consider the propagation of a plane wave between two parallel perfectly conducting planes.




Perspective view




Side view

Combining Equations [ VIII- 13c ] and [ VIII- 14c ], the electric field strength of the TE wave in the region between the plates may be written

$$\vec{\mathbf{E}} = \hat{\mathbf{y}}\left[E^{inc} \exp\left(i x\, k_1 \cos\theta_{inc}\right) + E^{ref} \exp\left(-i x\, k_1 \cos\theta_{inc}\right)\right] \exp\left(-i\, z\, k_1 \sin\theta_{inc}\right) \quad [\text{ VIII-27 }]$$

At $x = 0$ the field parallel to the surface of a perfect conductor must be zero so that $E^{ref} = -E^{inc}$ and, therefore,

$$\begin{aligned}\vec{\mathbf{E}} &= \hat{\mathbf{y}}\, E^{inc}\left[\exp\left(i x\, k_1 \cos\theta_{inc}\right) - \exp\left(-i\, x k_1 \cos\theta_{inc}\right)\right] \exp\left(-i z\, k_1 \sin\theta_{inc}\right) \\ &= \hat{\mathbf{y}}\, 2\, i\, E^{inc} \sin\left(x k_1 \cos\theta_{inc}\right) \exp\left(-i\, z\, \beta\right)\end{aligned} \quad [\text{ VIII-28 }]$$

where $\beta = k_1 \sin\theta_{inc}$. At the upper surface -- *i.e.* $x = d$ -- the field parallel to the surface of a perfect conductor must also be zero so that

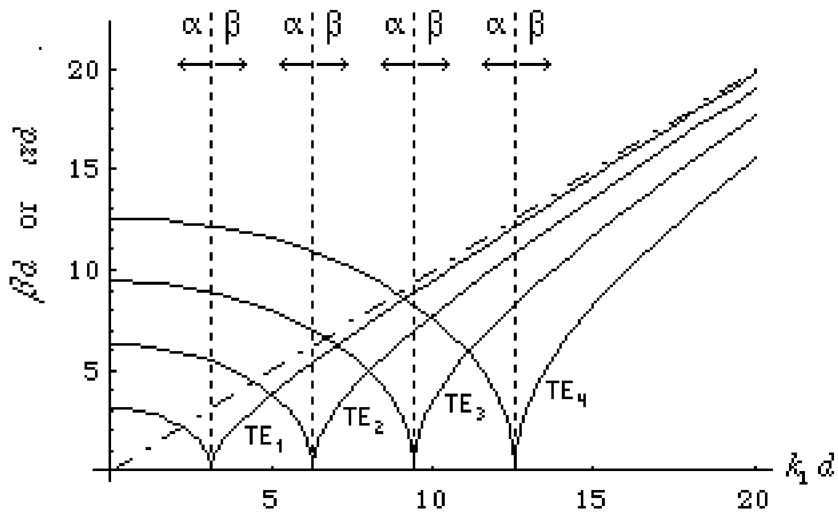
$$d\,k_1 \cos\theta_{inc} = n\,\pi \qquad \text{where } n = 1,2,3,\ldots \tag{VIII-29}$$

and, therefore,

$$\beta_{TE_n} = k_1 \sin\theta_{inc} = \sqrt{k_1^2 - k_1^2 \cos^2\theta_{inc}} = \sqrt{k_1^2 - (n\,\pi/d)^2} \qquad \text{where } n = 1,2,3,\ldots \tag{VIII-30}$$

which is **the dispersion relationship for TE waves in a parallel plate waveguide** with "cutoff" frequencies at

$$\omega_n^{cutoff} = (n\,\pi/d)\left(\sqrt{\mu_0\,\varepsilon_1}\right)^{-1} \qquad \text{where } n = 1,2,3,\ldots \tag{VIII-31}$$




Again combining Equations [ VIII- 13c ] and [ VIII- 14c ], the electric field strength of the TM wave in the region between the plates may be written

$$\vec{\mathbf{E}}_{||} = \hat{\mathbf{z}}\left(\eta_1 \cos\theta_{\text{inc}}\right)\left\{\mathrm{H}_{||}^{\text{inc}}\ \exp\left(i\, x\, k_1 \cos\theta_{\text{inc}}\right) - \mathrm{H}_{||}^{\text{ref}}\ \exp\left(-i\, x\, k_1 \cos\theta_{\text{inc}}\right)\right\} \exp\left(-i\, z\, k_1 \sin\theta_{\text{inc}}\right)$$
$$+\hat{\mathbf{x}}\left(\eta_1 \sin\theta_{\text{inc}}\right)\left\{\mathrm{H}_{||}^{\text{inc}}\ \exp\left(i x\ k_1 \cos\theta_{\text{inc}}\right) + \mathrm{H}_{||}^{\text{ref}}\ \exp\left(-i x\ k_1 \cos\theta_{\text{inc}}\right)\right\} \exp\left(-i z\ k_1 \sin\theta_{\text{inc}}\right) \quad \text{[ VIII-32 ]}$$

At $x = 0$ the field parallel to the surface of a perfect conductor must be zero so that $\mathrm{H}_{||}^{\text{ref}} = \mathrm{H}_{||}^{\text{inc}}$ and, therefore,

$$\vec{\mathbf{E}}_{||} = \hat{\mathbf{z}}\, 2\, i\, \mathrm{H}_{||}^{\text{inc}}\left(\eta_1 \cos\theta_{\text{inc}}\right) \sin\left(x\ k_1 \cos\theta_{\text{inc}}\right) \exp\left(-i z\ \beta\right)$$
$$+\hat{\mathbf{x}}\, 2\, \mathrm{H}_{||}^{\text{inc}}\left(\eta_1 \sin\theta_{\text{inc}}\right) \cos\left(x\ k_1 \cos\theta_{\text{inc}}\right) \exp\left(-i z\ \beta\right) \quad \text{[ VIII-33 ]}$$

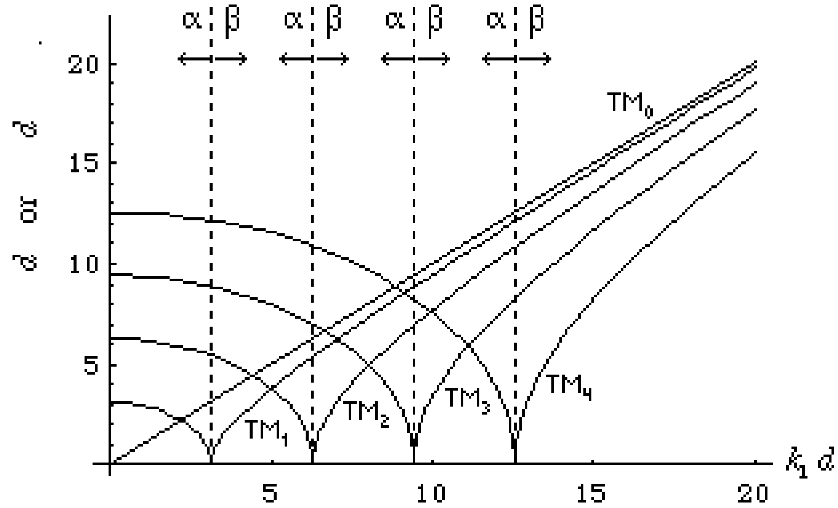
where $\beta = k_1 \sin\theta_{\text{inc}}$. At the upper surface -- *i.e.* $x = d$ -- again the field parallel to the surface of a perfect conductor must also be zero so that

$$d\, k_1 \cos\theta_{\text{inc}} = n\, \pi \qquad \text{where}\ n = 0,1,2,3,\ldots \qquad \text{[ VIII-34 ]}$$

and, therefore,

$$\beta_{\text{TM}_n} = k_1 \sin\theta_{\text{inc}} = \sqrt{k_1^2 - k_1^2 \cos^2\theta_{\text{inc}}} = \sqrt{k_1^2 - \left(n\, \pi / d\right)^2} \qquad \text{where}\ n = 0,1,2,3,\ldots \quad \text{[ VIII-35 ]}$$

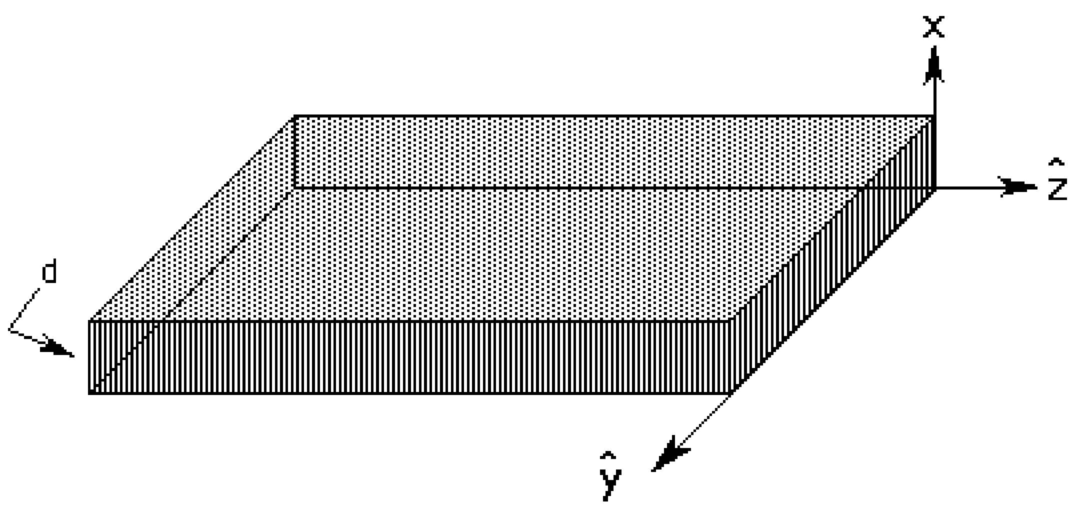
which is **the dispersion relationship for TM waves in a parallel plate waveguide.**



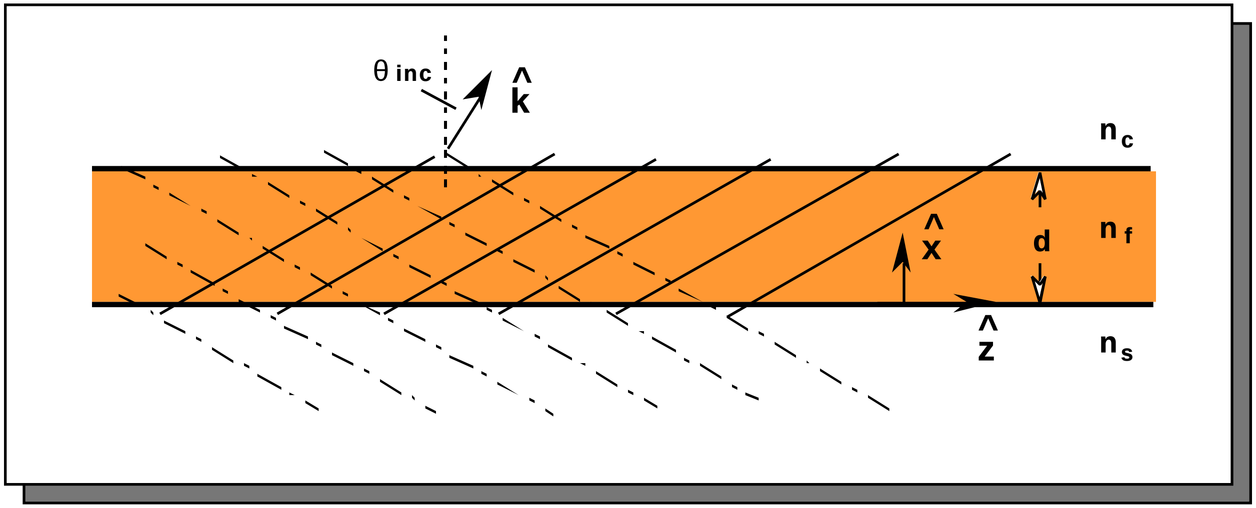

**Note that the $TM_0$ mode is a bona fide mode of propagation which does not have a "cutoff" frequency!**

## DIELECTRIC SLAB WAVEGUIDE:

Consider the propagation of waves "trap in" or "guided by" a dielectric slab of thickness *d*.



In its full generality this is moderately complicated problem, but a rather simple ray optics model of the propagation is sufficient to yield dispersion relationships for the various possible modes of propagation. To obtain such relationships, consider the total internal reflection of a sequence of plane waves as illustrated below.




In order for the multiplely reflected wave to be **self-consistence** the following, relatively obvious, phase condition must hold:[33]

$$\Delta\varphi_{x=d} + \Delta\varphi_{x=0} + 2\, k_1\, d \cos\theta_{\text{inc}} = m\, 2\pi \quad \text{where} \quad m = 0,1,2,3,\ldots \qquad [\text{ VIII- 36 }]$$

where $\Delta\varphi_{x=d}$ and $\Delta\varphi_{x=0}$ are, respectively, the phase shifts associated with the reflections at the upper and lower dielectric boundaries.

For **TE-modes of propagation** Equation [ VIII- 26a ] gives the phase shift at the boundary (called in the trade *the TE Goos-Hänchen shift*) and Equation [ VIII- 36 ] becomes

33 This equation is a direct generalization of Equations [ V-29 ] and [ V-34 ] which figured in our analysis of parallel plane waveguides.

$$k_f\, d\cos\theta_{inc} = m\pi + \tan^{-1}\left[\frac{\sqrt{\sin^2\theta_{inc} - \left(k_c/k_f\right)^2}}{\cos\theta_{inc}}\right] + \tan^{-1}\left[\frac{\sqrt{\sin^2\theta_{inc} - \left(k_s/k_f\right)^2}}{\cos\theta_{inc}}\right] \qquad \text{[ VIII- 37a ]}$$

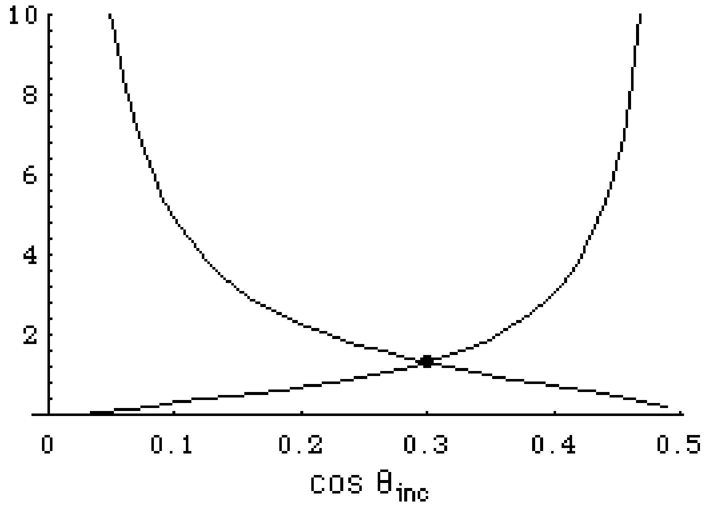
or

$$\left(\frac{2\pi d}{\lambda_0}\right)\sqrt{n_f^2 - n^2} = m\pi + \tan^{-1}\left[\frac{\sqrt{n^2 - n_c^2}}{\sqrt{n_f^2 - n^2}}\right] + \tan^{-1}\left[\frac{\sqrt{n^2 - n_s^2}}{\sqrt{n_f^2 - n^2}}\right] \qquad \text{[ VIII- 37b ]}$$

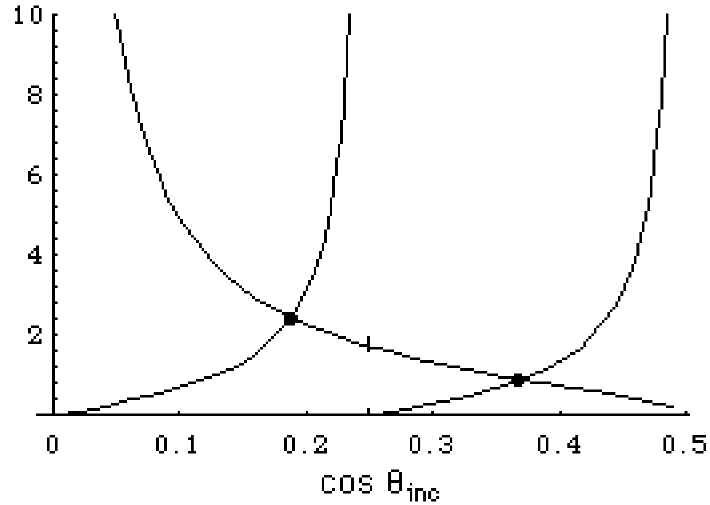
where $\sin\theta_{inc} \equiv \beta/k_f = n/n_f$ (***n* is the effective index of the propagation mode**). For the symmetric case (i.e., $n_c \equiv n_s$), the **self-consistence relationship for the TE modes** is given by

$$\frac{\sqrt{\sin^2\theta_{int} - \left(n_s/n_f\right)^2}}{\cos\theta_{int}} = \tan\left[\frac{n_f\, k_0\, d\cos\theta_{int}}{2} - m\frac{\pi}{2}\right] \qquad \text{[ VIII- 38 ]}$$

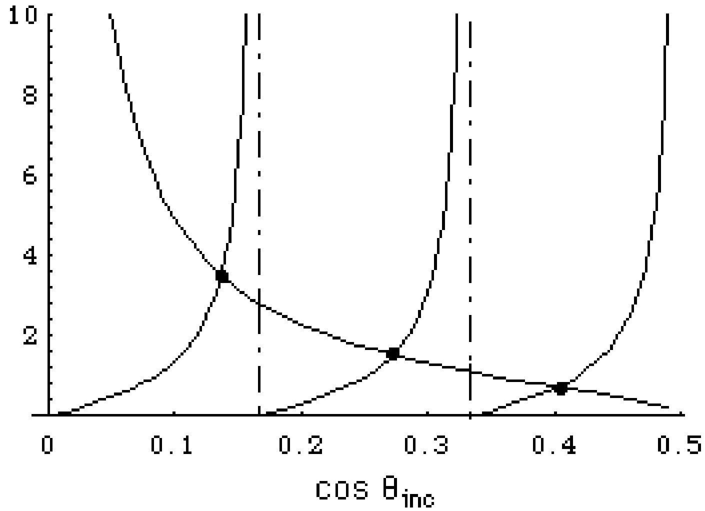
where $k_0 = \omega/c = 2\pi/\lambda_0$. This is a transcendental equation in the single variable $\cos\theta_{inc}$. Its solutions yield the allowed bounce angles, $\left(\theta_{inc}\right)_m$, of possible modes and, hence, the allowed propagation constants since $\beta = k_f \sin\theta_{inc}$ The left and right sides of this equation may be plot as a function of $\cos\theta_{inc}$ with $n_f\, k_0\, d = n_f\, 2\pi\left(d/\lambda_0\right)$ and $\sin\theta_{inc}^{crit} = n_s/n_f$ as a parameters. The intersections of such curves yield the allowed bounce angles as illustrated below



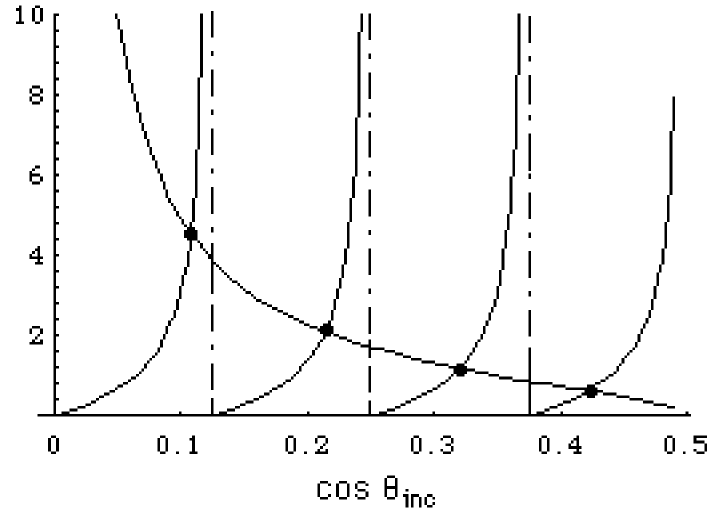

LHS and RHS of Equation [ VIII- 38 ]
for $n_1\left(d/\lambda_0\right) = 0.5$ and $\cos\theta_{\text{inc}}^{\text{crit}} = 0.5$




LHS and RHS of Equation [ VIII- 38 ]
for $n_1\left(d/\lambda_0\right) = 1.0$ and $\cos\theta_{\text{inc}}^{\text{crit}} = 0.5$




LHS and RHS of Equation [ VIII- 38 ]
for $n_1\left(d/\lambda_0\right) = 1.5$ and $\cos\theta_{\text{inc}}^{\text{crit}} = 0.5$




LHS and RHS of Equation [ VIII- 38 ]
for $n_1\left(d/\lambda_0\right) = 2.0$ and $\cos\theta_{\text{inc}}^{\text{crit}} = 0.5$

1974, Kogelnik and Ramaswamy [34] developed a convenient formalism for treating slab-waveguide problems. First they introduced three new waveguide parameter -- *viz*.

The normalized frequency/slab thickness parameter $V \equiv k_0\, d\, \sqrt{n_f^2 - n_s^2}$ [ VIII-39a ]

The normalized waveguide index parameter $b \equiv \left(n^2 - n_f^2\right)/\left(n_f^2 - n_s^2\right)$ [ VIII-39b ]

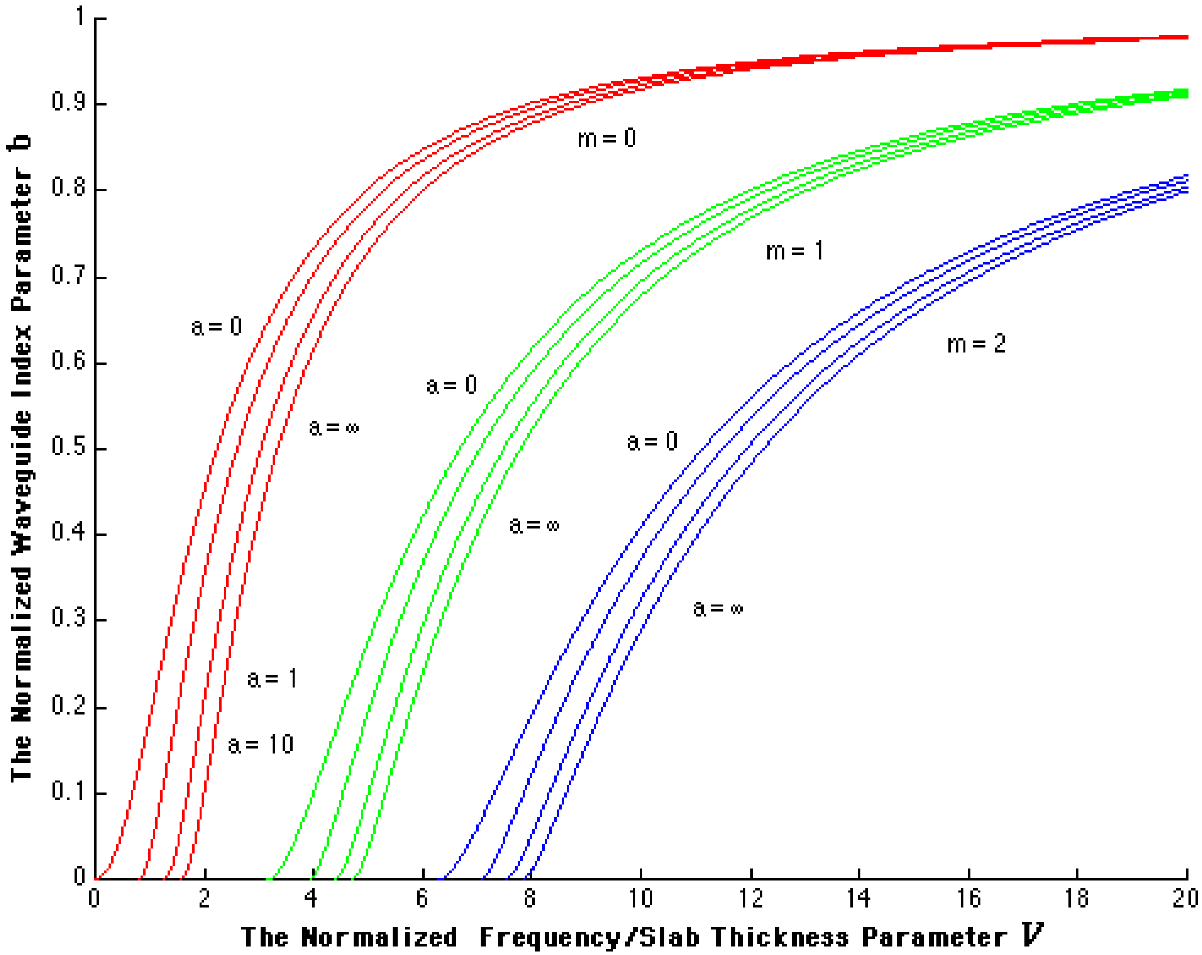
The normalized waveguide asymmetry parameter $a \equiv \left(n_s^2 - n_c^2\right)/\left(n_f^2 - n_s^2\right)$ [ VIII-39c ]

They then showed that Equation [ VIII-37b ] could be written

$$V\sqrt{1-b} = m\,\pi + \tan^{-1}\sqrt{b/(1-b)} + \tan^{-1}\sqrt{(a+b)/(1-b)} \qquad \text{[ VIII-39c ]}$$

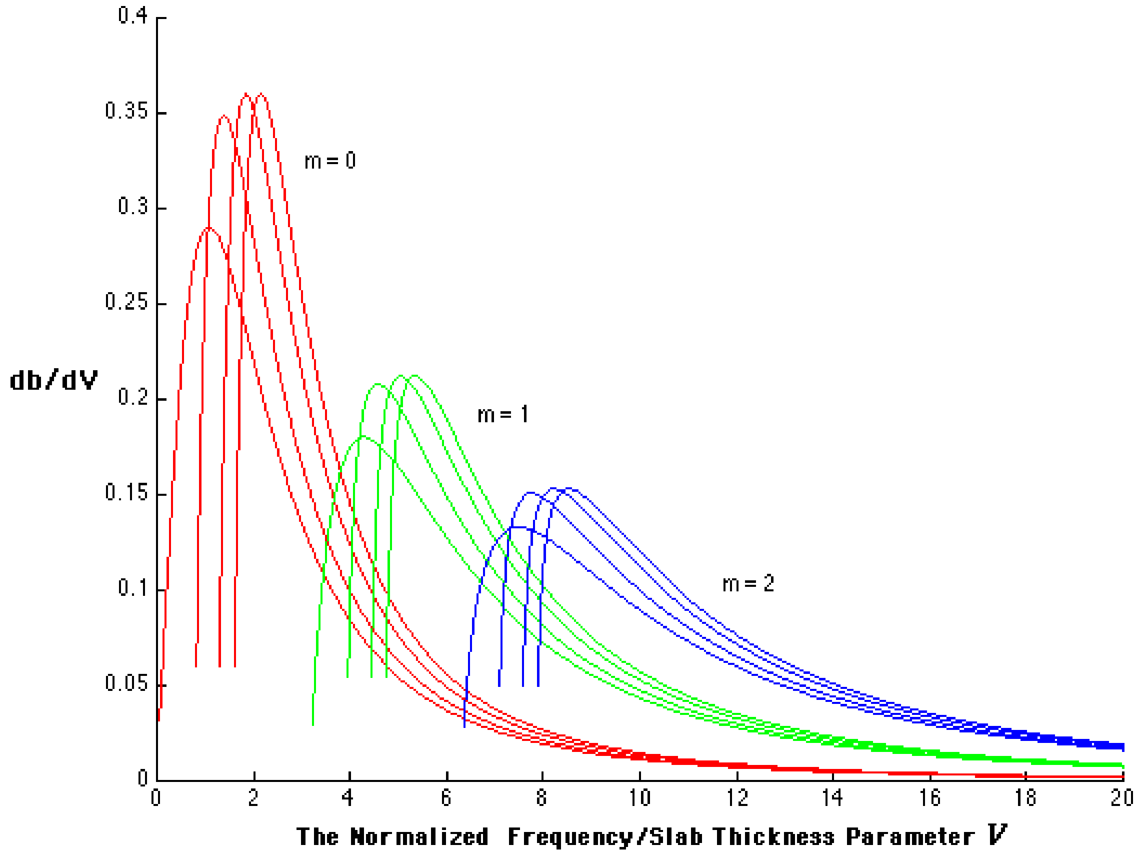
which can be used to generate the following family of curves:

[34] H. Kogelnik and V. Ramaswamy, Appl. Opt. **13**, 1857 (1974).




It is also useful to differentiate Equation [ VIII-39c ] to obtain

$$\frac{db}{dV} = 2(1-b)\left[\sqrt{1/b} + \sqrt{1/(a+b)}\right]^{-1} \qquad \text{[ VIII-40 ]}$$



0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
db/dV
m = 0
m = 1
m = 2
0
2
4
6
8
10
12
14
16
18
20
The Normalized Frequency/Slab Thickness Parameter V