V. PLANE WAVE PROPAGATION IN A LINEAR, HOMOGENEOUS, ANISOTROPIC DIELECTRIC MEDIA ("CRYSTAL OPTICS"):

Our objective here is to formulate a general approach to the subject of wave propagation in anisotropic dielectrics which makes use of ideas familiar from other branches of mathematical physics -- *viz.* the "eigenvalue problem.".[18] For reasons that will soon become abundantly clear, treatments of "crystal optics" focus on the behavior of the dielectric displacement vector, $\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)$ rather than on the electric field vector.[19] For non-magnetic dielectrics the components of the dielectric displacement are usefully represented as the Cartesian coordinates coordinates of figure called the "ellipsoid of wave normals," the "optical indicatix," the "index ellipsoid" or the "reciprocal ellipsoid."

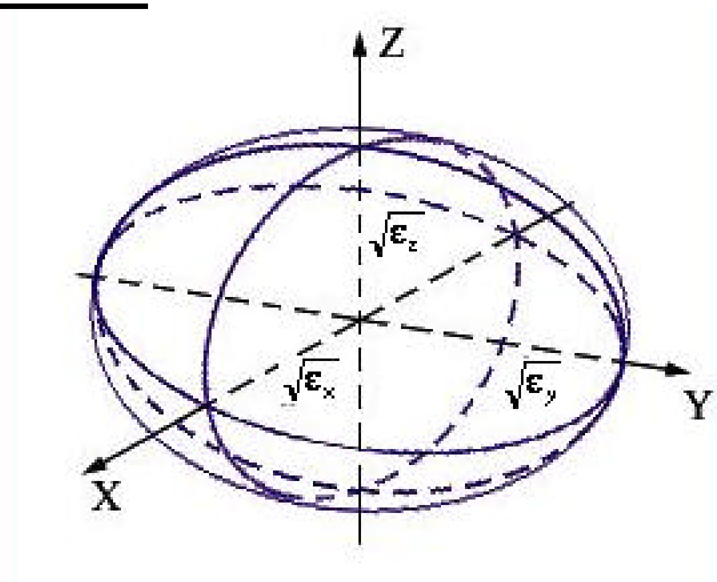
Since the stored electrical energy is given by

$$U_e = (1/2)\,\vec{\mathbf{D}}\ \vec{\mathbf{E}} = (1/2)\,\vec{\mathbf{D}}\ \vec{\varepsilon}^{\,-1}\ \vec{\mathbf{D}}$$

we can write in the *principle axis system*

$$\frac{\left(\dfrac{D_x}{\sqrt{2U_e}}\right)^2}{\varepsilon_x} + \frac{\left(\dfrac{D_y}{\sqrt{2U_e}}\right)^2}{\varepsilon_y} + \frac{\left(\dfrac{D_z}{\sqrt{2U_e}}\right)^2}{\varepsilon_z} = 1$$

where $\{\varepsilon_x,\varepsilon_y,\varepsilon_z\}$ are *principle axis* values of the dielectric constant tensor.



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[18] Kaiser S. Kunz in 1977 presented a similar treatment of this problem in a paper entitiled "Treatment of optical propagation in crystals using projection dyadics," Am. J. Phys., **Vol. 45**, 1977, pp. 267-269.

[19] Perhaps the most authoritative treatment of "Crystal Optics" is found in Max Born and Emil Wolf, *Principle of Optics*, Pergamon Press (1986), Chapter 14.

Now let us first combine Equations [I-8a] and [I-8b] to obtain a generalized Helmholz equation for a homogeneous anisotropic dielectric

$$\vec{\nabla}\times\left(\vec{\nabla}\times\left[\overleftrightarrow{\varepsilon}^{-1}(\omega)\cdot\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)\right]\right) = \omega^2\mu_o\,\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega) \qquad \text{[V-1]}$$

In general, we would hope to be able to find a set of ***eigenmodes*** of the homogeneous problem that would satisfy the scalar eigenequation

$$\nabla^2\,\mathrm{D}^{(\sigma)}(\vec{\mathbf{r}},\omega) + n_{(\sigma)}^2\,k_0^2\,\mathrm{D}^{(\sigma)}(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[V-2]}$$

where $n_{(\sigma)}$ is the effective index of refraction of the σ-th eigenmode. We can be quite specific for the case of **plane wave eigenmodes** where we suppose that all fields have a spatial dependence $\exp(\mp i\,\vec{\mathbf{k}}\cdot\vec{\mathbf{r}})$. From Equation [I-8c] we see that $\vec{\mathbf{D}}(\vec{\mathbf{r}},\omega)$ must, in general, be orthogonal to $\vec{\mathbf{k}}$ so that we may write

$$\vec{\mathbf{D}}\,(\vec{\mathbf{k}},\omega) = \mathrm{D}^{(1)}(\vec{\mathbf{k}},\omega)\,\hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}},\omega) + \mathrm{D}^{(2)}(\vec{\mathbf{k}},\omega)\,\hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}},\omega) \qquad \text{[V-3]}$$

where the $\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})$'s are polarization unit vectors which are orthogonal to $\breve{\mathbf{k}}$, the unit vector parallel to $\vec{\mathbf{k}}$. Since the components of $\overleftrightarrow{\varepsilon}$ in the general case may be complex -- *e.g.,* in the case of magneto-optical media -- the eigenmodes may be polarized along "complex directions" -- *e.g.,* “screw axes” defining the sense of circular polarization -- and we must use great care in all vector manipulations. Thus, we define a set of adjoint or conjugate unit vectors by means of the relationships

$$\breve{\mathbf{k}}\cdot\hat{\mathbf{k}} = 1; \qquad \breve{\mathbf{k}}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}}) = 0; \qquad \breve{\mathbf{t}}^{(\sigma)}(\hat{\mathbf{k}})\cdot\hat{\mathbf{k}} = 0; \qquad \breve{\mathbf{t}}^{(\sigma)}(\hat{\mathbf{k}})\cdot\hat{\mathbf{t}}^{(\sigma')}(\breve{\mathbf{k}}) = \delta_{\sigma\sigma'} \qquad \text{[V-4]}$$

Thus, with $\vec{\mathbf{k}} = \mathrm{k}\,\hat{\mathbf{k}} = \mathrm{k}^*\,\breve{\mathbf{k}}$ (so that $\vec{\mathbf{k}}\cdot\vec{\mathbf{k}} = |\mathrm{k}|^2$) the representation for $\vec{\mathbf{D}}(\vec{\mathbf{k}})$ in Equation [V-3] automatically satisfies Equation. [I-8c]. A key problem is the representation of the ubiquitous vector operation

$$\overleftrightarrow{\ }(\vec{\mathbf{r}})\ \vec{\mathbf{F}}(\vec{\mathbf{r}}) = \vec{\ } \times\left(\vec{\ } \times \vec{\mathbf{F}}(\vec{\mathbf{r}})\right) = \vec{\ }\ \vec{\ }\ \ \vec{\mathbf{F}}(\vec{\mathbf{r}}) - \ {}^2\vec{\mathbf{F}}(\vec{\mathbf{r}}) \qquad [\text{ V-5 }]$$

which appears in Equation [V-1]. For plane wave representation of the operator becomes

$$\overleftrightarrow{\ }(\vec{\mathbf{k}}) \equiv \left(\vec{\mathbf{k}}\ \ \vec{\mathbf{k}}\right)\overleftrightarrow{\mathbf{1}} - \vec{\mathbf{k}}\,\vec{\mathbf{k}} = |\mathrm{k}|^2\left\{\overleftrightarrow{\mathbf{1}} - \breve{\mathbf{k}}\,\hat{\mathbf{k}}\right\}. \qquad [\text{ V-6 }]$$

Since

$$\overleftrightarrow{\mathbf{1}} = \hat{\mathbf{k}}\ \breve{\mathbf{k}} + \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\ \breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}}) + \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\ \breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}}) \qquad [\text{ V-7 }]$$

the plane wave representation of the operator simplifies to

$$\overleftrightarrow{\ }(\vec{\mathbf{k}}) = |\mathrm{k}|^2\left\{\hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\ \breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}}) + \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\ \breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}})\right\} \qquad [\text{ V-8}]$$

Using this representation of the $\overleftrightarrow{\ }(\vec{\mathbf{k}})$ operator and the representation for $\vec{\mathbf{D}}(\vec{\mathbf{k}})$ in Equation. [V-1], the generalized Helmholz equation -- *i.e.* Equation [V-1] -- can be written

$$\begin{aligned}&|\mathrm{k}|^2\{\mathrm{D}^{(1)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\left[\breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\right] + \mathrm{D}^{(1)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\left[\breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\right] + \\ &\mathrm{D}^{(2)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}})\left[\breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\right] + \mathrm{D}^{(2)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\left[\breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\right]\} \qquad [\text{ V-9 }] \\ &= \omega^2\mu_0\left\{\mathrm{D}^{(1)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}}) + \mathrm{D}^{(2)}(\vec{\mathbf{k}})\ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}})\right\}\end{aligned}$$

<u>**Crucial point**</u>**:** To obtain an eigenvalue equation we need to choose the eigenvectors $\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})$ so that

$$\breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}}) = \breve{\mathbf{t}}^{(2)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}}) \equiv 0 \qquad [\text{ V-10 }]$$

If we can find eigenvectors defined in this way, then Equation [I-9] becomes an eigenvalue equation with eigenvalues -- *i.e.*, the inverse refractive indices -- given by

$$\left[n_1(\vec{\mathbf{k}})\right]^{-2} = \mathrm{k}_0^2\big/|\mathrm{k}|^2 = \varepsilon_0\ \breve{\mathbf{t}}^{(1)}(\hat{\mathbf{k}})\ \ \overleftrightarrow{\varepsilon}^{-1}\ \ \hat{\mathbf{t}}^{(1)}(\breve{\mathbf{k}}) \qquad [\text{ V-11a }]$$

$$\left[n_2(\vec{\mathbf{k}})\right]^{-2} = \mathrm{k}_0^2 \big/ |\mathrm{k}|^2 = \varepsilon_0 \; \breve{\mathbf{t}}^{(2)}(\widehat{\mathbf{k}}) \; \overleftrightarrow{\varepsilon}^{-1} \; \widehat{\mathbf{t}}^{(2)}(\breve{\mathbf{k}}) \qquad [\text{ V-11b }]$$

These results -- *i.e.*, Equations [I-10] and [I-11] -- are a complete formal solution of the problem. However, they are difficult to apply in the general case and an addition relationship -- *viz.*, the Fresnel equation of wave normals -- is found to be extremely useful as the starting point for actual computations. For plane waves, Equation [V-1] can be rewritten as

$$\left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right] \; \vec{\mathbf{E}}\left(\vec{\mathbf{k}}\right) = \vec{\mathbf{k}} \, \vec{\mathbf{k}} \; \vec{\mathbf{E}}\left(\vec{\mathbf{k}}\right) \qquad [\text{ V-12 }]$$

Multiplying this equation through by $\vec{\mathbf{k}} \; \left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right]^{-1}$ we obtain

$$\vec{\mathbf{k}} \; \vec{\mathbf{E}}\left(\vec{\mathbf{k}}\right) = \left\{\vec{\mathbf{k}} \; \left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right]^{-1} \; \vec{\mathbf{k}}\right\} \; \vec{\mathbf{k}} \; \vec{\mathbf{E}}\left(\vec{\mathbf{k}}\right) \qquad [\text{ V-13a }]$$

or

$$1 = \vec{\mathbf{k}} \; \left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right]^{-1} \; \vec{\mathbf{k}} \qquad [\text{ V-13b }]$$

From this result we may develop two important relationships. Using the principal axes coordinates of the dielectric tensor, we can write

$$\left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right] = \sum_{\lambda = a}^{c} \left(|\mathrm{k}|^2 - \left(\mathrm{k}_0^2/\varepsilon_0\right) \varepsilon_{(\lambda)}\right) \widehat{\mathbf{a}}_\lambda \; \breve{\mathbf{a}}_\lambda \qquad [\text{ V-14a }]$$

and

$$\left[|\mathrm{k}|^2 \; \overleftrightarrow{\mathbf{1}} - \left(\mathrm{k}_0^2/\varepsilon_0\right) \overleftrightarrow{\varepsilon}\right]^{-1} = \sum_{\lambda = a}^{c} \left(|\mathrm{k}|^2 - \left(\mathrm{k}_0^2/\varepsilon_0\right) \varepsilon_{(\lambda)}\right)^{-1} \widehat{\mathbf{a}}_\lambda \; \breve{\mathbf{a}}_\lambda \qquad [\text{ V-14b }]$$

Therefore, Equation [I-14b] becomes

$$\frac{\left(\widehat{\mathbf{k}} \; \breve{\mathbf{a}}_a\right)\left(\breve{\mathbf{k}} \; \widehat{\mathbf{a}}_a\right)}{n^2 - n_a^2} + \frac{\left(\widehat{\mathbf{k}} \; \breve{\mathbf{a}}_b\right)\left(\breve{\mathbf{k}} \; \widehat{\mathbf{a}}_b\right)}{n^2 - n_b^2} + \frac{\left(\widehat{\mathbf{k}} \; \breve{\mathbf{a}}_c\right)\left(\breve{\mathbf{k}} \; \widehat{\mathbf{a}}_c\right)}{n^2 - n_c^2} = \frac{1}{n^2} \qquad [\text{ V-15a }]$$

where $n_\lambda^2 = \varepsilon_{(\lambda)}/\varepsilon_0$. Since $\sum_{\lambda=a}^{c} \hat{\mathbf{a}}_\lambda \; \breve{\mathbf{a}}_\lambda = \overleftrightarrow{\mathbf{1}}$ (or $\sum_{\lambda=a}^{c} \left(\breve{\mathbf{k}} \;\; \hat{\mathbf{a}}_\lambda\right)\left(\breve{\mathbf{a}}_\lambda \;\; \hat{\mathbf{k}}\right) = 1$) we may also write Equation [V-15a] as

$$\frac{\left(\hat{\mathbf{k}} \;\; \breve{\mathbf{a}}_a\right)\left(\breve{\mathbf{k}} \;\; \hat{\mathbf{a}}_a\right)}{\dfrac{1}{n^2} - \dfrac{1}{n_a^2}} + \frac{\left(\hat{\mathbf{k}} \;\; \breve{\mathbf{a}}_b\right)\left(\breve{\mathbf{k}} \;\; \hat{\mathbf{a}}_b\right)}{\dfrac{1}{n^2} - \dfrac{1}{n_b^2}} + \frac{\left(\hat{\mathbf{k}} \;\; \breve{\mathbf{a}}_c\right)\left(\breve{\mathbf{k}} \;\; \hat{\mathbf{a}}_c\right)}{\dfrac{1}{n^2} - \dfrac{1}{n_c^2}} = 0 \qquad \text{[V-15b]}$$

This latter expression is the famous ***Fresnel equation of wave normals***.[20]

APPLICATIONS OF THE FORMAL SOLUTION:

Uniaxial Dielectric Crystals:

For an optical material with uniaxial symmetry, the inverse dielectric tensor in the principal axes system must have the form [21]

$$\overline{\overline{\varepsilon}}^{-1} = \varepsilon^{-1}\left(\hat{\mathbf{a}}_a \, \hat{\mathbf{a}}_a + \hat{\mathbf{a}}_b \, \hat{\mathbf{a}}_b\right) + \varepsilon_\parallel^{-1} \, \hat{\mathbf{a}}_c \, \hat{\mathbf{a}}_c \; . \qquad \text{[V-16]}$$

Thus, Equation [I-19b] becomes

$$\left[\frac{1}{n^2} - \frac{1}{n^2}\right]\left\{\left[\frac{1}{n^2} - \frac{1}{n_\parallel^2}\right]\sin^2\vartheta + \left[\frac{1}{n^2} - \frac{1}{n^2}\right]\cos^2\vartheta\right\} = 0 \qquad \text{[V-17]}$$

so that

$$n_o^{-2} = n^{-2}; \quad \text{and} \quad n_e^{-2} = n^{-2}\cos^2\vartheta + n_\parallel^{-2}\sin^2\vartheta \qquad \text{[V-18]}$$

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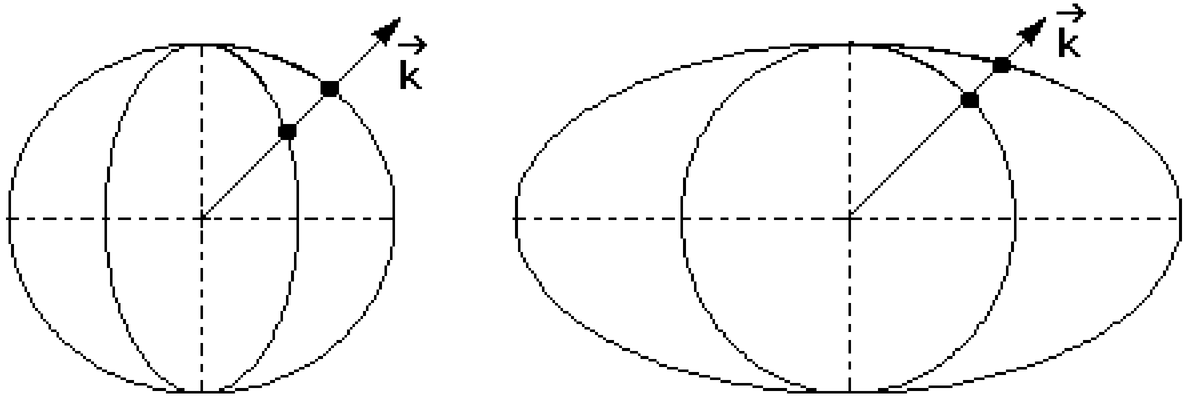
20 In its commonly used form, the Fresnel equation becomes
$$\frac{\left(\hat{\mathbf{k}}\right)_x^2}{\dfrac{1}{n^2} - \dfrac{1}{n_x^2}} + \frac{\left(\hat{\mathbf{k}}\right)_y^2}{\dfrac{1}{n^2} - \dfrac{1}{n_y^2}} + \frac{\left(\hat{\mathbf{k}}\right)_z^2}{\dfrac{1}{n^2} - \dfrac{1}{n_z^2}} = 0$$

21 In this instance there is no need to trouble ourselves about conjugate unit vectors.

where $\hat{\mathbf{k}} \cdot \hat{\mathbf{a}}_c = \cos\vartheta$. The subscript **"o"** identifies the "ordinary" mode and the subscript **"e"** the "extraordinary" mode. These results are usually plotted as follows:



where the intersections of the $\vec{\mathbf{k}}$ vector yield the "ordinary" and "extraordinary" velocities of propagation for a given $\vec{\mathbf{k}}$.

Further, if we take

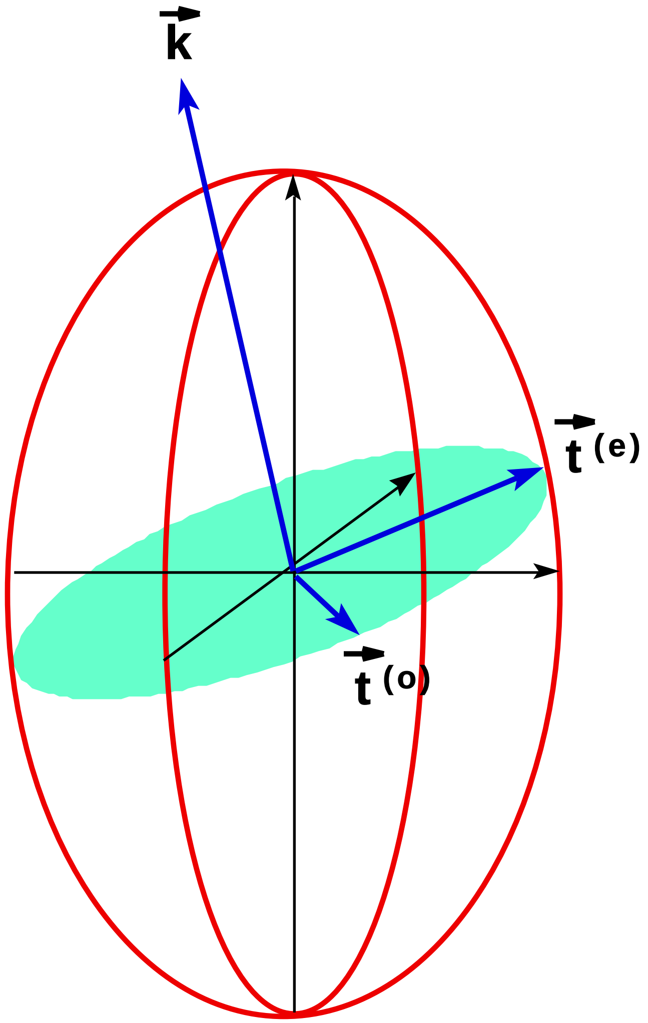
$$\hat{\mathbf{k}} = \sin\phi\ \sin\vartheta\ \hat{\mathbf{a}}_a + \cos\phi\ \sin\vartheta\ \hat{\mathbf{a}}_b + \cos\vartheta\ \hat{\mathbf{a}}_c \qquad \text{[V-19}$$

it is a *bagatelle* to show that

$$\widehat{\mathbf{t}}^{(o)}(\breve{\mathbf{k}}) = \breve{\mathbf{t}}^{(o)}(\widehat{\mathbf{k}}) = \cos\phi\ \hat{\mathbf{a}}_a - \sin\phi\ \hat{\mathbf{a}}_b \qquad \text{[V-20a]}$$

$$\widehat{\mathbf{t}}^{(e)}(\breve{\mathbf{k}}) = \breve{\mathbf{t}}^{(e)}(\widehat{\mathbf{k}}) = \cos\vartheta\ \left(\cos\phi\ \hat{\mathbf{a}}_a + \sin\phi\ \hat{\mathbf{a}}_b\right) - \sin\vartheta\ \hat{\mathbf{a}}_c \qquad \text{[V-20b]}$$

and that these quations are consistent with Equations [V-10] and [V-11].



Magneto-optical Media:

For a simple magneto-optical substance we may write the dielectric dyadic in the form [22]

[22] See, for example, Section 82 in L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press (1960).

$$\overleftrightarrow{\varepsilon}^{-1} = \left[\overline{\overline{\mathbf{1}}} + i\,\gamma\left(\hat{\mathbf{a}}_a\,\hat{\mathbf{a}}_b - \hat{\mathbf{a}}_b\,\hat{\mathbf{a}}_a\right)\right]\varepsilon^{-1} \qquad \text{[V-21]}$$

If we introduce the conjugate principal axes

$$\begin{aligned} \hat{\mathbf{a}}_+ &= \breve{\mathbf{a}}_- = \left(1/\sqrt{2}\right)\left(\hat{\mathbf{a}}_a + i\,\hat{\mathbf{a}}_b\right) \\ \hat{\mathbf{a}}_- &= \breve{\mathbf{a}}_+ = \left(1/\sqrt{2}\right)\left(\hat{\mathbf{a}}_a - i\,\hat{\mathbf{a}}_b\right) \\ \hat{\mathbf{a}}_{\|} &= \breve{\mathbf{a}}_{\|} = \hat{\mathbf{a}}_c \end{aligned} \qquad \text{[V-22]}$$

we obtain the dielectric dyadic in the so called ***normal form*** -- *viz*.

$$\overleftrightarrow{\varepsilon}^{-1} = \left[\left(1-\gamma\right)\hat{\mathbf{a}}_+\,\breve{\mathbf{a}}_+ + \left(1+\gamma\right)\hat{\mathbf{a}}_-\,\breve{\mathbf{a}}_- + \hat{\mathbf{a}}_{\|}\,\breve{\mathbf{a}}_{\|}\right]\varepsilon^{-1} \qquad \text{[V-23]}$$

Again from Equation [V-15b] it is trivial to show that

$$n_{\pm}^{-2} = \left[1 \pm \gamma\,\cos\vartheta\right]\varepsilon^{-1} \qquad \text{[V-24]}$$

Using the resolution of $\hat{\mathbf{k}}$ as given in Equation [V-4] we may show that

$$\hat{\mathbf{t}}^{\pm}(\breve{\mathbf{k}}) = \left(\frac{i}{2}\right)\left\{\exp\left(-i\,\phi\right)\left[1 \mp \cos\vartheta\right]\hat{\mathbf{a}}_+ - \exp\left(i\,\phi\right)\left[1 \pm \cos\vartheta\right]\hat{\mathbf{a}}_- \pm \sqrt{2}\,\sin\vartheta\;\hat{\mathbf{a}}_{\|}\right\} \qquad \text{[V-25a]}$$

and

$$\breve{\mathbf{t}}^{\pm}(\hat{\mathbf{k}}) = \left(\frac{i}{2}\right)\left\{-\exp\left(i\,\phi\right)\left[1 \mp \cos\vartheta\right]\breve{\mathbf{a}}_+ + \exp\left(-i\,\phi\right)\left[1 \pm \cos\vartheta\right]\breve{\mathbf{a}}_- \mp \sqrt{2}\,\sin\vartheta\;\breve{\mathbf{a}}_{\|}\right\} \qquad \text{[V-25b]}$$

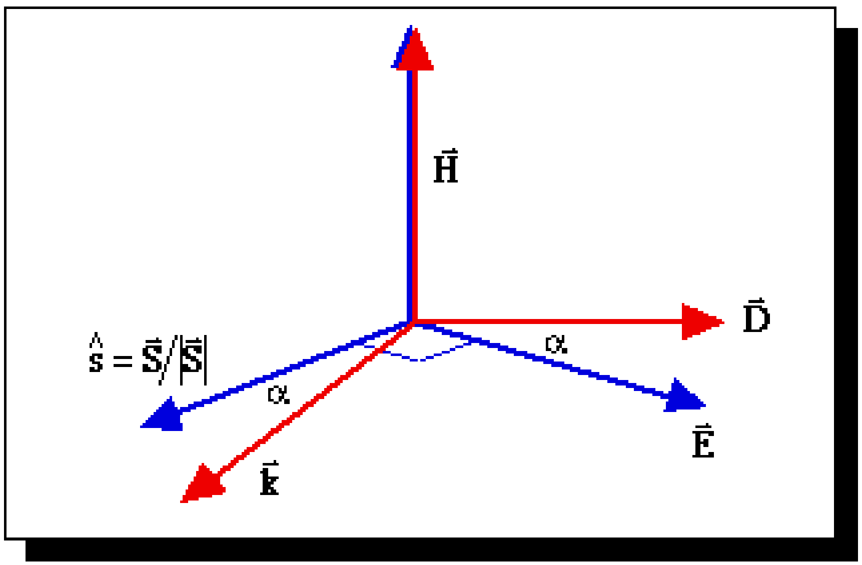
ENERGY FLOW IN ANISOTROPIC MEDIA

As previously noted, the content of Equations [I-10] and [I-11] represents in some sense a complete formal solution of the wave propagation problem. However, from a practical point of view it is essential to consider how energy propagates in anisotropic

media. To that end, we note that the time averaged Poynting vector associated with a given plane wave-- *viz*.

$$\vec{\mathbf{S}}\left(\vec{\mathbf{k}},\omega\right)=\frac{1}{2}\,\vec{\mathbf{E}}\left(\vec{\mathbf{k}},\omega\right)\times\vec{\mathbf{H}}^{*}\left(\vec{\mathbf{k}},\omega\right) \qquad \text{[V-26]}$$

propagates in a direction $\hat{\mathbf{s}}\left(\vec{\mathbf{k}},\omega\right)$ -- conventionally designated the "ray" direction -- which is orthogonal to both $\vec{\mathbf{E}}\left(\vec{\mathbf{k}},\omega\right)$ and $\vec{\mathbf{H}}\left(\vec{\mathbf{k}},\omega\right)$ as shown below.



The time averaged total stored energy is given by

$$\begin{aligned} U\left(\vec{\mathbf{k}},\omega\right) &= \frac{1}{4}\left[\vec{\mathbf{E}}\left(\vec{\mathbf{k}},\omega\right)\ \vec{\mathbf{D}}^{*}\left(\vec{\mathbf{k}},\omega\right)+\vec{\mathbf{B}}\left(\vec{\mathbf{k}},\omega\right)\ \vec{\mathbf{H}}^{*}\left(\vec{\mathbf{k}},\omega\right)\right] \\ &= \frac{\left|\vec{\mathbf{k}}\right|}{2\,\omega}\,\hat{\mathbf{k}}\ \ \vec{\mathbf{E}}\left(\vec{\mathbf{k}},\omega\right)\times\vec{\mathbf{H}}^{*}\left(\vec{\mathbf{k}},\omega\right)=\frac{\left|\vec{\mathbf{k}}\right|}{\omega}\,\hat{\mathbf{k}}\ \ \vec{\mathbf{S}}\left(\vec{\mathbf{k}},\omega\right) \end{aligned} \qquad \text{[V-27]}$$

and, thus, we see that the "ray" or "energy flow" velocity, v_{ray}, for a given $\hat{\mathbf{s}}\left(\vec{\mathbf{k}},\omega\right)$ is given by

$$\frac{1}{v_{ray}}=\frac{\left|\vec{\mathbf{k}}\right|}{\omega}\,\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}\left(\hat{\mathbf{k}},\omega\right)=\frac{1}{v_{phase}}\,\hat{\mathbf{k}}\cdot\hat{\mathbf{s}}\left(\hat{\mathbf{k}},\omega\right) \qquad \text{[V-28]}$$

We write the time averaged Poynting vector associated with a given eigenmode as

$$\begin{aligned}\vec{\mathbf{S}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)&=\frac{1}{2}\,\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)\times\vec{\mathbf{H}}^{(\sigma)*}\left(\vec{\mathbf{k}}\right)=\frac{1}{2\,\omega\,\mu_0}\,\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)\times\left[\vec{\mathbf{k}}\times\vec{\mathbf{E}}^{(\sigma)*}\left(\vec{\mathbf{k}}\right)\right]\\&=\frac{1}{2\,\omega\,\mu_0}\left\{\vec{\mathbf{k}}\left|\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)\right|^2-\vec{\mathbf{E}}^{(\sigma)*}\left(\vec{\mathbf{k}},\omega\right)\left[\vec{\mathbf{k}}\cdot\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)\right]\right\}\end{aligned} \qquad \text{[V-29]}$$

where $\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)$ is the electric field associated with the eigenmode. Using Equations [V-3], [V-10] and [V-11] this field can be expressed as

$$\begin{aligned}\vec{\mathbf{E}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)&=\left\{\left[\breve{\mathbf{t}}^{(\sigma)}(\hat{\mathbf{k}})\cdot\vec{\vec{\varepsilon}}^{\,-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})+\left[\hat{\mathbf{k}}\cdot\vec{\vec{\varepsilon}}^{\,-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]\hat{\mathbf{k}}\right\}D^{(\sigma)}(\vec{\mathbf{k}})\\&=\frac{1}{\varepsilon_0\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^2}\left\{\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})+\varepsilon_0\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^2\left[\hat{\mathbf{k}}\cdot\vec{\vec{\varepsilon}}^{\,-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]\hat{\mathbf{k}}\right\}D^{(\sigma)}(\vec{\mathbf{k}})\\&=\frac{1}{\varepsilon_0\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^2}\left\{\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})+g^{(\sigma)}(\breve{\mathbf{k}})\,\hat{\mathbf{k}}\right\}D^{(\sigma)}(\vec{\mathbf{k}})\end{aligned} \qquad \text{[V-30]}$$

where $g^{(\sigma)}(\breve{\mathbf{k}})=\varepsilon_0\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^2\left[\hat{\mathbf{k}}\cdot\vec{\vec{\varepsilon}}^{\,-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]=\dfrac{\left[\hat{\mathbf{k}}\cdot\vec{\vec{\varepsilon}}^{\,-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]}{\left[\breve{\mathbf{t}}^{(\sigma)}(\hat{\mathbf{k}})\cdot\vec{\vec{\varepsilon}}^{\,-1}\cdot\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\right]}$. Using this parameterization, the modal Poynting vector can be expressed as

$$\vec{\mathbf{S}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)=\frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{\left|D^{(\sigma)}(\vec{\mathbf{k}})/\varepsilon_0\right|^2}{\left[n_{(\sigma)}(\vec{\mathbf{k}})\right]^3}\left\{\hat{\mathbf{k}}-\hat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\,g^{(\sigma)}(\breve{\mathbf{k}})\right\} \qquad \text{[V-31]}$$

and the associated ray vector as

$$\hat{\mathbf{s}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)=\frac{\widehat{\mathbf{k}}-\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\,g^{(\sigma)}(\breve{\mathbf{k}})}{\sqrt{\left|\widehat{\mathbf{k}}-\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\,g^{(\sigma)}(\breve{\mathbf{k}})\right|}}=\frac{\widehat{\mathbf{k}}-\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})\,g^{(\sigma)}(\breve{\mathbf{k}})}{\sqrt{1+\left|g^{(\sigma)}(\breve{\mathbf{k}})\right|^{2}}} \qquad [\text{ V-32 }]$$

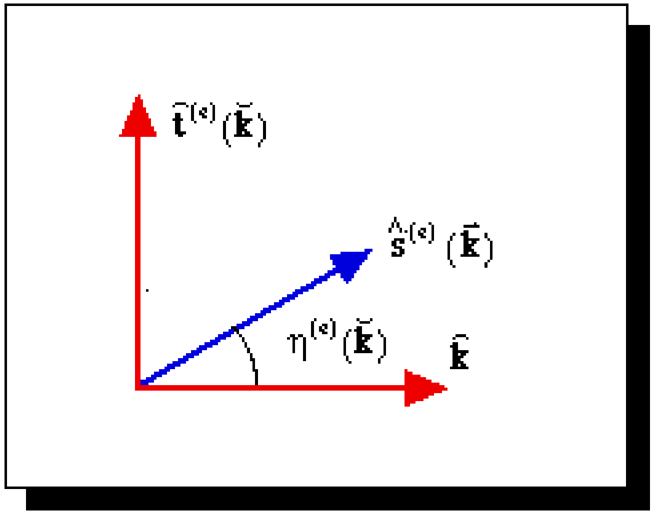
If we take $\widehat{\mathbf{k}}$ and $\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})$ as reference directions, the ray vector, $\hat{\mathbf{s}}^{(\sigma)}\left(\vec{\mathbf{k}}\right)$, lies in the plane containing $\widehat{\mathbf{k}}$ and $\widehat{\mathbf{t}}^{(\sigma)}(\breve{\mathbf{k}})$ at an angle $\eta^{(\sigma)}(\breve{\mathbf{k}})=-\tan^{-1}\left[g^{(\sigma)}(\breve{\mathbf{k}})\right]$ with respect to the direction $\widehat{\mathbf{k}}$.

For an optical material with uniaxial symmetry, we may use Equations [V-16], [V-19] and [V-20] to evaluate $g^{(\sigma)}(\breve{\mathbf{k}})$. In particular, we may easily see that for the ordinary mode

$$\tan\eta^{(o)}(\breve{\mathbf{k}})=-g^{(o)}(\breve{\mathbf{k}})=0 \qquad [\text{ V-33a }]$$

or $\hat{\mathbf{s}}^{(o)}\left(\vec{\mathbf{k}}\right)=\widehat{\mathbf{k}}$ and for the extraordinary mode

$$\tan\eta^{(e)}(\breve{\mathbf{k}})=-g^{(e)}(\breve{\mathbf{k}})=\frac{\sin\vartheta\,\cos\vartheta\,\left(\varepsilon^{-1}-\varepsilon_{\|}^{-1}\right)}{\varepsilon^{-1}\cos^{2}\vartheta+\varepsilon_{\|}^{-1}\sin^{2}\vartheta}\,. \qquad [\text{ V-33b }]$$


$\hat{t}^{(\epsilon)}(\vec{k})$
$\hat{s}^{(\epsilon)}(\vec{k})$
$\eta^{(\epsilon)}(\vec{k})$
$\hat{k}$