# III. THE PARAXIAL WAVE EQUATION -- PROPAGATION OF GAUSSIAN BEAMS IN UNIFORM MEDIA

## DERIVATION OF PARAXIAL WAVE EQUATION:

In point-to-point communication, we may think of the electromagnetic field as propagating in a kind of "searchlight" mode -- *i.e.* a beam of finite width that propagates in some particular direction. In analyzing this mode of wave propagation, we make use of an important solution to the so call paraxial approximation of the electromagnetic wave equation (or, more precisely, the paraxial approximation of the Helmholz equation).

To that end, we first derive the paraxial approximation and then examine the free-space ***Gaussian Beam*** solution(s). We start with the homogeneous Helmholz equation for the vector potential in the form -- see Equation [ I-13a ]

$$ {}^{2}\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)+\omega^{2}\,\mu_{0}\,\varepsilon(\omega)\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)= {}^{2}\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)+k^{2}\,\vec{\mathbf{A}}(\vec{\mathbf{r}},\omega)=0 \qquad \text{[ III-1 ]}$$

We are looking for a wave propagating in, say, the z-direction, so we write a particular component of the potential in the form

$$ \mathrm{A}_{\alpha}(\vec{\mathbf{r}},\omega)=\Psi(\vec{\mathbf{r}},\omega)\exp(-ikz) \qquad \text{[ III-2 ]}$$

The function $\Psi(\vec{\mathbf{r}},\omega)$ represents a spatial modulation or "masking" of a plane wave propagating in the z-direction. The z-direction is obviously special and it is useful to appropriately parse the differential operators. For the **grad** operator we may write

$$ grad\{\qquad\}=\vec{}\{\qquad\}=\vec{}_{t}\{\qquad\}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\{\qquad\} \qquad \text{[ III-3 ]}$$

where, for example,

$$\vec{\nabla}_t \{ \qquad \} = \hat{\mathbf{x}} \frac{\partial}{\partial x} \{ \qquad \} + \hat{\mathbf{y}} \frac{\partial}{\partial y} \{ \qquad \} . \qquad \text{[ III-4 ]}$$

so that

$$\vec{\nabla} A_\alpha(\vec{\mathbf{r}},\omega) = \left[ \vec{\nabla}_t \Psi(\vec{\mathbf{r}},\omega) + \hat{\mathbf{z}} \frac{\partial}{\partial z} \Psi(\vec{\mathbf{r}},\omega) - ik\, \hat{\mathbf{z}}\, \Psi(\vec{\mathbf{r}},\omega) \right] \exp(-ik\, z) \qquad \text{[ III-5 ]}$$

For the **Laplacian** operator we may write

$$\nabla^2 A_\alpha(\vec{\mathbf{r}},\omega) = \nabla_t^2 \Psi(\vec{\mathbf{r}},\omega) \exp(-ik\, z) + \frac{\partial}{\partial z} \left\{ \left[ \frac{\partial}{\partial z} \Psi(\vec{\mathbf{r}},\omega) - ik\, \Psi(\vec{\mathbf{r}},\omega) \right] \exp(-ikz) \right\} \qquad \text{[ III-6 ]}$$

where, for example,

$$\nabla_t^2 \{ \qquad \} = \frac{\partial^2}{\partial x^2} \{ \qquad \} + \frac{\partial^2}{\partial y^2} \{ \qquad \} \qquad \text{[ III-7 ]}$$

Therefore,

$$\nabla^2 A_\alpha(\vec{\mathbf{r}},\omega) = \left[ \nabla_t^2 \Psi(\vec{\mathbf{r}},\omega) + \frac{\partial^2}{\partial z^2} \Psi(\vec{\mathbf{r}},\omega) - 2ik \frac{\partial}{\partial z} \Psi(\vec{\mathbf{r}},\omega) - k^2 \Psi(\vec{\mathbf{r}},\omega) \right] \exp(-ikz) \qquad \text{[ III-8 ]}$$

and the parsed Helmholz equation (**without approximation**) becomes

$$\nabla_t^2 \Psi(\vec{\mathbf{r}},\omega) - 2ik \frac{\partial}{\partial z} \Psi(\vec{\mathbf{r}},\omega) + \frac{\partial^2}{\partial z^2} \Psi(\vec{\mathbf{r}},\omega) = 0 \qquad \text{[ III-9 ]}$$

The **paraxial approximation** is precisely defined by the condition

$$2ik \frac{\partial}{\partial z} \Psi(\vec{\mathbf{r}},\omega) >> \frac{\partial^2}{\partial z^2} \Psi(\vec{\mathbf{r}},\omega) \qquad \text{[ III-10 ]}$$

which means that the longitudinal variation in the modulation function, $\Psi(\vec{\mathbf{r}},\omega)$, changes very little in the wavelength associated with beam -- *i.e.* $2\pi/k$. In this approximation, we neglect the third term and obtain the equation

$$\boxed{\nabla_t^2\,\Psi(\vec{\mathbf{r}},\omega) - 2ik\,\frac{\partial}{\partial z}\,\Psi(\vec{\mathbf{r}},\omega) = 0} \qquad \text{[ III-11 ]}$$

which is called the **paraxial approximation** of the wave equation.[15]

## SOLUTIONS OF THE PARAXIAL WAVE EQUATION

### The Gaussian beam

To inform or motivate our next step, we consider the paraxial approximation of a known solution of the Helmholz equation -- *i.e.* a spherical wave

$$\frac{\exp(-ikr)}{r} = \frac{\exp\left(-ik\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}} = \frac{\exp\left(-ikz\sqrt{1+\dfrac{x^2+y^2}{z^2}}\right)}{z\sqrt{1+\dfrac{x^2+y^2}{z^2}}} \approx \frac{\exp(-ikz)\exp\left[\dfrac{-ik\left(x^2+y^2\right)}{2z}\right]}{z} \qquad \text{[ III-12 ]}$$

Reflecting on the "quadratic" form of this approximate expression, it is reasonable to look for an axially symmetric solution of the paraxial wave equation in the following form -- *i.e.* a **Gaussian beam**:

$$\Psi_G(\rho,z,\omega) = A_G\,\exp\left[-i\,P(z)\right]\,\exp\left[-\frac{ik\,\rho^2}{2q(z)}\right] \qquad \text{[ III-13 ]}$$

15 Obvious the paraxial equation has the same mathematical form as the Schrödinger equation and, thus, all that is know about solutions of that equation may be directly applied to understand issues in light propagation (or visa versa).

where $\rho^2 = x^2 + y^2$ .

We test our conjecture by substituting the Gaussian beam function -- *i.e.* Equation [ III-13 ] -- into the paraxial wave equation -- *i.e.* Equation [ III-11 ] -- to wit

$$\exp\left[-i\,P(z)\right]\ \nabla_t^2 \left\{ \exp\left[-\frac{ik\,\rho^2}{2q(z)}\right]\right\} - 2ik\,\frac{\partial}{\partial z}\left\{\exp\left[-iP(z)\right]\exp\left[-\frac{ik\,\rho^2}{2q(z)}\right]\right\} = 0 \ . \qquad [\ \text{III-14}\ ]$$

Executing the indicated operations, we obtain

$$\exp\left[-i\,P(z)\right]\exp\left[-\frac{ik\,\rho^2}{2q(z)}\right]\left\{-\frac{2\,ik}{q(z)} - \frac{k^2\rho^2}{\left[q(z)\right]^2} - 2ik\left[-i\frac{\partial}{\partial z}P(z) + \frac{ik\,\rho^2}{2\left[q(z)\right]^2}\frac{\partial}{\partial z}q(z)\right]\right\} = 0 \qquad [\ \text{III-15a}\ ]$$

or simplifying

$$2\,k\frac{\partial}{\partial z}P(z) + \frac{2ik}{q(z)} + \frac{k^2\rho^2}{\left[q(z)\right]^2}\left\{1 - \frac{\partial}{\partial z}q(z)\right\} = 0 \ . \qquad [\ \text{III-15b}\ ]$$

Hence, for an **arbitrary** $\rho$ this equation is separable into two parts -- *viz*.

$$\frac{k^2\rho^2}{\left[q(z)\right]^2}\left\{1 - \frac{\partial}{\partial z}q(z)\right\} = 0 \longrightarrow \frac{\partial}{\partial z}q(z) = 1 \qquad [\ \text{III-15c}\ ]$$

and

$$2\,k\frac{\partial}{\partial z}P(z) + \frac{2ik}{q(z)} = 0 \longrightarrow \frac{\partial}{\partial z}P(z) = \frac{-i}{q(z)} \qquad [\ \text{III-15d}\ ]$$

which are satisfied by the simple solutions

$$q(z) = z + q_0 \qquad [\ \text{III-16a}\ ]$$

and

$$\frac{\partial}{\partial z} P(z) = \frac{-i}{z+q_0} = -i \frac{\partial}{\partial z} \ln\left[z+q_0\right] \longrightarrow P(z) = -i \ln\left[z+q_0\right] . \qquad [\text{ III-16b }]$$

On comparison with the paraxial approximation of a spherical wave -- *i.e* Equation [ III-12 ] -- we may write $q(z)$ in terms of a radius of curvature $R(z)$ and a width $w(z)$ -- *viz.*

$$\frac{1}{q(z)} = \frac{1}{z+q_0} = \frac{1}{R(z)} + \frac{-i\,2}{k\, w^2(z)} . \qquad [\text{ III-17 }]$$

To standardize the constants of integration we assume a **plane wavefront** at an arbitrary reference point $z = 0$ -- *i.e.* we take $R(0) \equiv \infty$. Thus,

$$\frac{1}{R(0)} = 0 \qquad [\text{ III-18a }]$$

and

$$\frac{-i\,2}{k\, w^2(0)} \equiv \frac{1}{q_0} \qquad q_0 = \frac{i\,k\,w^2(0)}{2} = \frac{i\,\pi\,w^2(0)}{\lambda} = i\,L_F \qquad [\text{ III-18b }]$$

where $L_F = k\, w^2(0)/2 = \pi\, w^2(0)/\lambda$ is the critical Gaussian beam scaling parameter which is called variously the **Fresnel length**, the **diffraction length**, or the **confocal parameter**. In terms of this parameter, Equation [ III-17 ] may be written

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{-i\,2}{k\, w^2(z)} = \frac{1}{z+i\,L_F} = \frac{z-i\,L_F}{z^2+L_F^2} \qquad [\text{ III-19 }]$$

Equating real and imaginary parts, we obtain

$$\frac{1}{R(z)} = \frac{z}{z^2+L_F^2} \qquad \text{and} \qquad \frac{-i\,2}{k\, w^2(z)} = \frac{-iL_F}{z^2+L_F^2}$$

or, finally, in standardized form

$$\boxed{\begin{aligned} R(z) &= z\left[1+L_{\mathrm{F}}^2/z^2\right] \\ w^2(z) &= w^2(0)\left[1+z^2/L_{\mathrm{F}}^2\right] \\ &\text{where}\quad L_{\mathrm{F}} = \pi\, w^2(0)/\lambda \end{aligned}} \qquad [\text{ III-20 }]$$

Now since Equation [ III-16b ] may be written

$$P(z) = -i\ \ln\left[z+q_0\right] = -i\ \ln\left[z+i\ L_{\mathrm{F}}\right] = -i\left\{\ln\left[z^2+L_{\mathrm{F}}^2\right] + i\ \tan^{-1}\left[L_{\mathrm{F}}/z\right]\right\}l \qquad [\text{ III-16b' }]$$

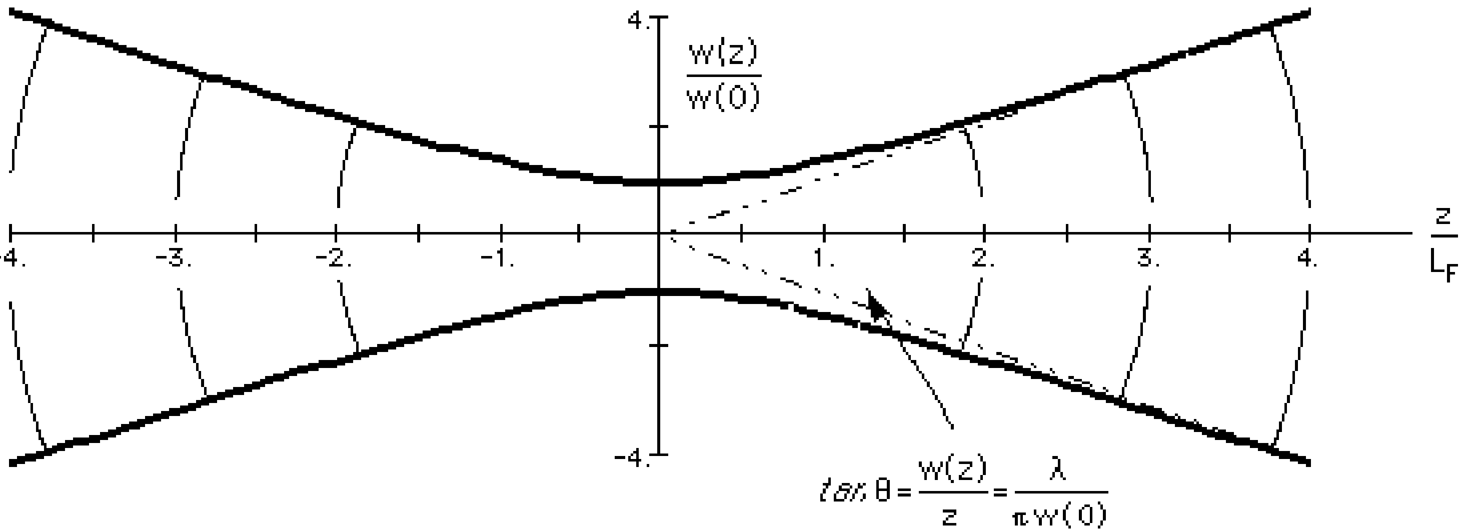
we may write

$$\exp\left[-i\ P(z)\right] = \frac{\exp\left[-i\ \tan^{-1}\left[L_{\mathrm{F}}/z\right]\right]}{\sqrt{z^2+L_{\mathrm{F}}^2}} = \frac{\exp\left[-i\ \tan^{-1}\left[L_{\mathrm{F}}/z\right]\right]}{z\sqrt{1+L_{\mathrm{F}}^2/z^2}}$$

to obtain the usual, **officially approved** form of the Gaussian Beam

$$\begin{aligned} \Psi_{\mathrm{G}}(\rho,z,\omega) &= A_{\mathrm{G}}\ \exp\left[-i\ P(z)\right]\exp\left[-\frac{ik\rho^2}{2\,q(z)}\right] \\ &= A_{\mathrm{G}}\ \frac{w(0)}{w(z)}\ \exp\left[-i\ \tan^{-1}\left[L_{\mathrm{F}}/z\right]\right]\exp\left[-\frac{ik\rho^2}{2\,R(z)}\right]\exp\left[-\frac{\rho^2}{w^2(z)}\right] \end{aligned} \qquad [\text{ III-21 }]$$

The following kind of picture is sometimes found to be helpful in understanding the propagation of a Gaussian Beam (the bold curve depicts the spatial variation of the beam width and the light curve the beam curvature at particular points in space):




**Higher order Hermite-Gaussian beams**

In order to study the propagation of higher order beams, we substitute the following trial solution:

$$\begin{aligned}\Psi_{H\text{-}G}(\rho, z, \omega) &= F(x, y, z)\, \Psi_G(\rho, z, \omega) \\ &= f\left(\frac{x}{w}\right) g\left(\frac{y}{w}\right) \exp\left[-i\, \Phi(z)\right] \Psi_G(\rho, z, \omega)\end{aligned} \qquad \text{[ III-22 ]}$$

into the paraxial wave equation -- *viz.* Equation [ III-11 ] -- and obtain

$$\begin{aligned}\mathrm{F}(x,y,z)\,\nabla_t^2\,\Psi_G(\rho,z,\omega)+2\left[\vec{\nabla}_t\mathrm{F}(x,y,z)\cdot\vec{\nabla}_t\Psi_G(\rho,z,\omega)\right]+\Psi_G(\rho,z,\omega)\,\nabla_t^2\,\mathrm{F}(x,y,z)\\ -2ik\,\Psi_G(\rho,z,\omega)\frac{\partial}{\partial z}\mathrm{F}(x,y,z)-2ik\,\mathrm{F}(x,y,z)\frac{\partial}{\partial z}\Psi_G(\rho,z,\omega)=0\end{aligned} \qquad [\text{ III-23 }]$$

Since the sum of the first and fifth terms already satisfies the paraxial wave equation, Equation [ III-23 ] reduces to

$$\frac{\mathrm{f}''}{\mathrm{f}}+2ik\left[\frac{dw}{dz}-\frac{w}{q}\right]x\,\frac{\mathrm{f}'}{\mathrm{f}}+\frac{\mathrm{g}''}{\mathrm{g}}+2ik\left[\frac{dw}{dz}-\frac{w}{q}\right]y\,\frac{\mathrm{g}'}{\mathrm{g}}-2\,k\,w^2\,\frac{d\Phi}{dz}=0 \qquad [\text{ III-24 }]$$

From Equations [ III-19 ] and [ III-20 ] we see that

$$\frac{dw}{dz}-\frac{w}{q}=\frac{w}{R}-\left(\frac{w}{R}+\frac{-i\,2}{k\,w}\right)=\frac{i\,2}{k\,w}$$

so that the reduced equation -- *i.e.* Equation [ III-24 ] -- becomes

$$\frac{\mathrm{f}''}{\mathrm{f}}-4\,\xi\,\frac{\mathrm{f}'}{\mathrm{f}}+\frac{\mathrm{g}''}{\mathrm{g}}-4\,\varsigma\,\frac{\mathrm{g}'}{\mathrm{g}}-2\,k\,w^2\,\frac{d\Phi}{dz}=0 \qquad [\text{ III-25 }]$$

where $\xi = x/w$ and $\varsigma = y/w$.

A **Hermite polynomial** of order n[16] has the following differential equation:

$$\frac{d^2}{d\tau^2}\mathrm{H}_n(\tau)-2\tau\,\frac{d}{d\tau}\mathrm{H}_n(\tau)+2n\,\mathrm{H}_n(\tau)=0. \qquad [\text{ III-26 }]$$

16 The Hermite polynomials have the generator $\mathrm{H}_n(\tau)=(-1)^n\,\exp(\tau^2)\,\frac{d^n}{d\tau^n}\exp(-\tau^2)$ .

With the simple change in variables $\tau = \sqrt{2}\,\xi = \sqrt{2}\; x/w$ and $\sigma = \sqrt{2}\,\varsigma = \sqrt{2}\; y/w$ Equation [ III-25 ] may be written

$$\frac{1}{f}\left[\frac{d^2 f}{d\tau^2} - 2\tau\,\frac{d f}{d\tau}\right] + \frac{1}{g}\left[\frac{d^2 g}{d\sigma^2} - 2\sigma\,\frac{d g}{d\sigma}\right] - 2\,kw^2\,\frac{d\Phi}{d z} = 0 \qquad \text{[ III-27 ]}$$

Thus, it is apparent that we can write the functions $f\left(\frac{x}{w}\right)$ and $g\left(\frac{y}{w}\right)$ as Hermite polynomials -- *viz.*

$$f\left(\frac{x}{w}\right) = H_n(\tau) = H_n\left(\sqrt{2}\;\frac{x}{w}\right) \quad \text{and} \quad g\left(\frac{x}{w}\right) = H_m(\sigma) = H_m\left(\sqrt{2}\;\frac{y}{w}\right)$$

if we require that $2\,kw^2\,\frac{d\Phi}{dz} = -2(n+m)$. Hence

$$\frac{d\Phi}{d z} = -\frac{(n+m)}{kw^2} = -\frac{(n+m)\,L_F}{2\left[L_F^2 + z^2\right]}$$

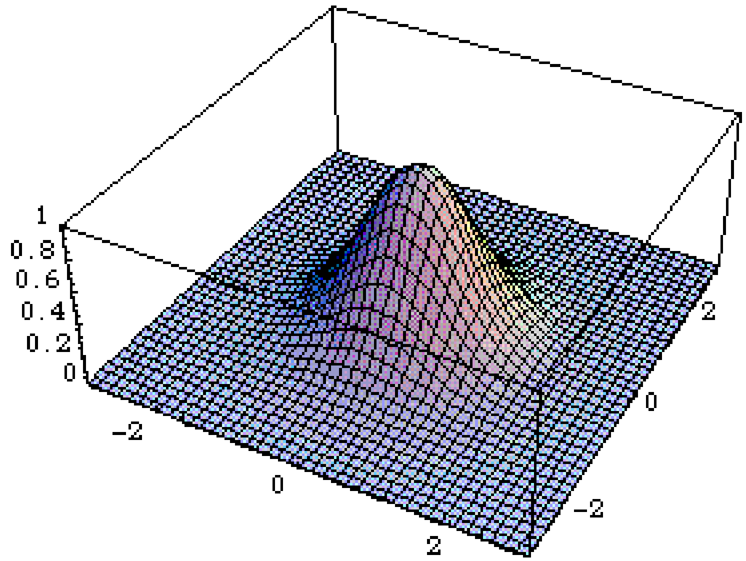
or

$$\Phi(z) = -(n+m)\; a\tan\left(z/L_F\right).$$

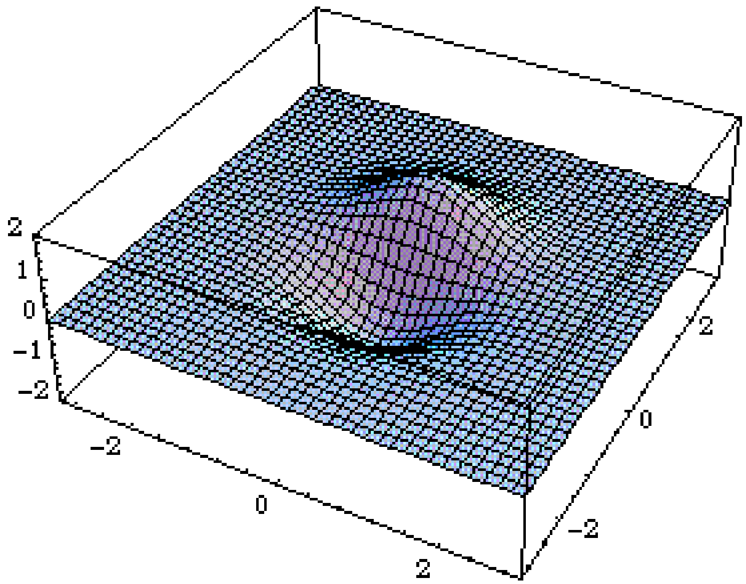
Finally we may write a general solution for the paraxial equation as

$$\begin{aligned}\Psi^{nm}_{H\text{-}G}(\rho, z, \omega) = A^{nm}_{H\text{-}G}\;\frac{w(0)}{w(z)}\; H_n\left(\sqrt{2}\;\frac{x}{w}\right) H_m\left(\sqrt{2}\;\frac{y}{w}\right) \\ \times \exp\left[i\,[n+m+1]\,\tan^{-1}\left(z/L_F\right)\right] \exp\left[-\frac{ik\rho^2}{2R(z)}\right] \exp\left[-\frac{\rho^2}{w^2(z)}\right]\end{aligned} \qquad \text{[ III-28 ]}$$

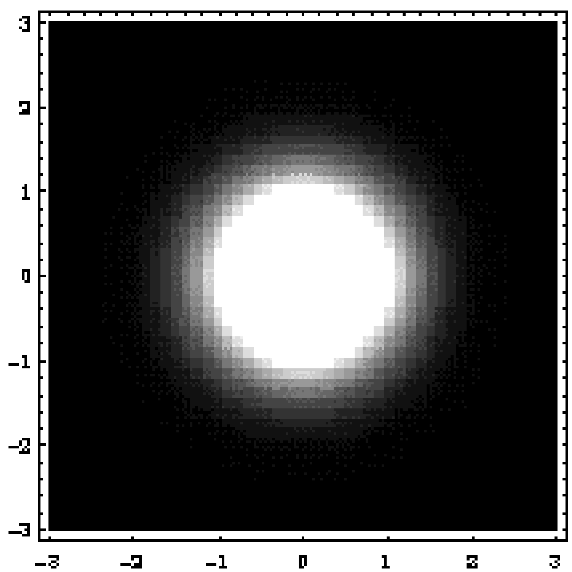
## A GALLERY OF HERMITE-GAUSSIAN FIELD DISTRIBUTIONS

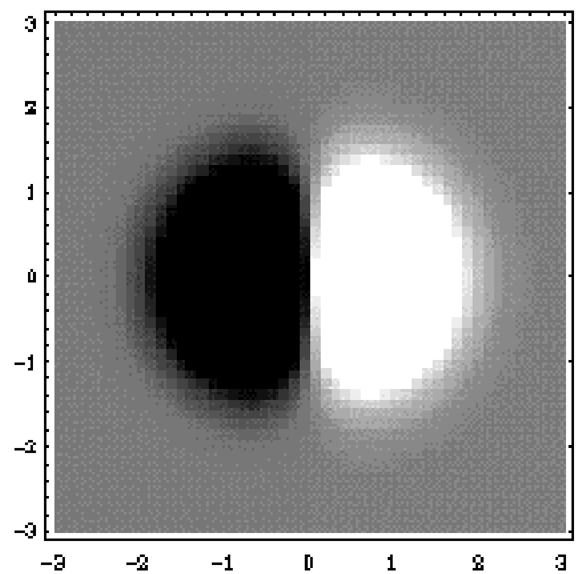


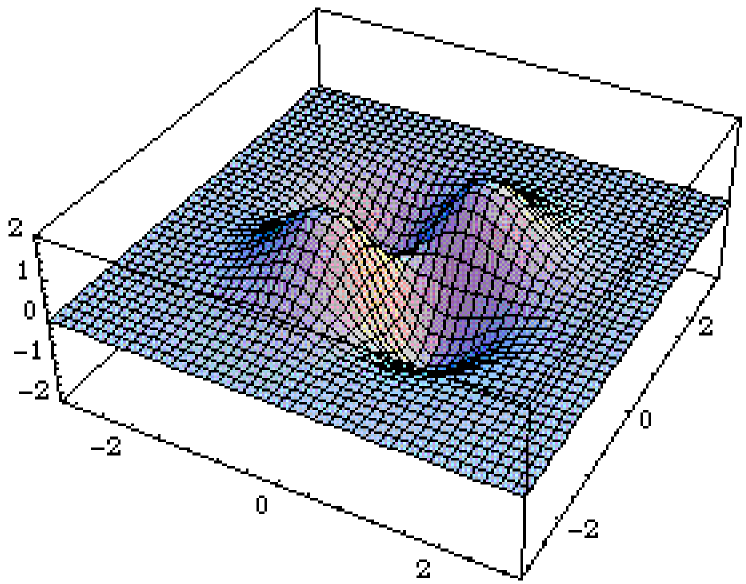

**[0, 0] Hermite-Gaussian**

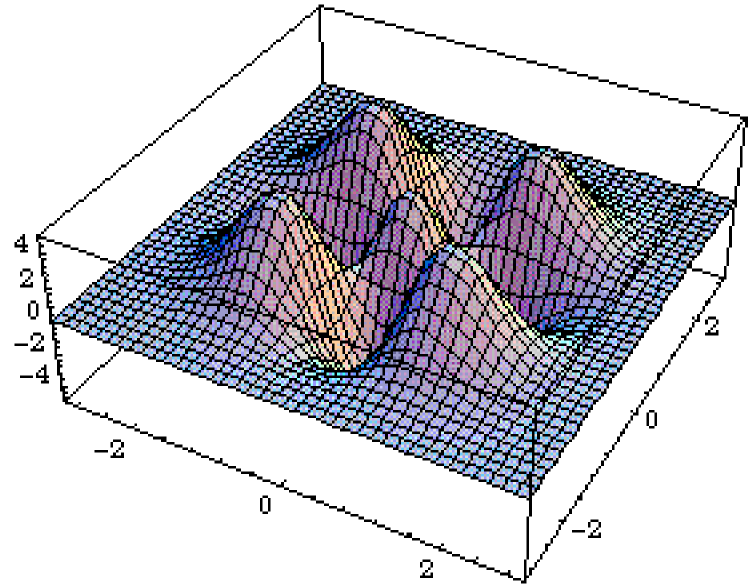



**[0, 1] Hermite-Gaussian**



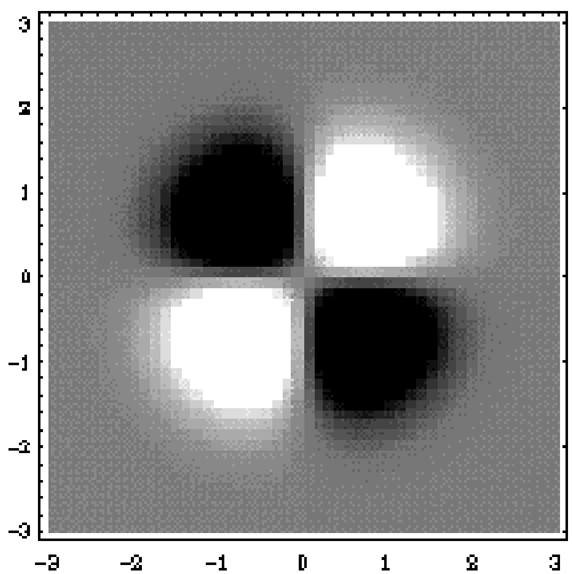



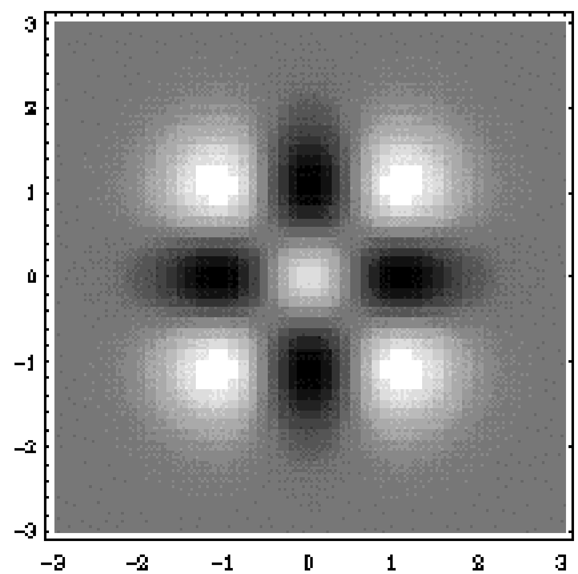



**[1, 1] Hermite-Gaussian**

**[2, 2] Hermite-Gaussian**



## <u>GAUSSIAN BEAM TRANSFORMATION MATRICES</u>

What we have shown above is that a given Hermite-Gaussian beam is essentially completely specified or defined by the complex function $q(z)$. In propagating through an optical system, the beams are transformed by various optical components. **<u>The amazing fact is that the transformation produced by a given component follows a simple ABCD transformation law</u>** -- *viz.*

$$\boxed{q_2 = \frac{Aq_1 + B}{Cq_1 + D}} \qquad \text{[ III-29 ]}$$

**<u>where A, B, C, D are the matrix elements found in our analysis of geometric optics!!</u>**

To **"prove"** this, we argue by example. For example, the transformation through a uniform dielectric region is given by

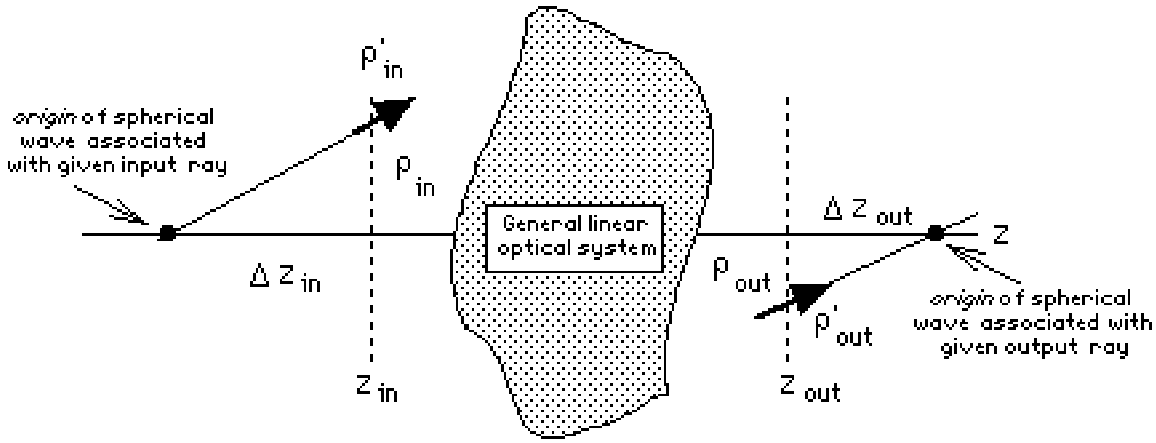
$$q_2 = q_1 + L$$

so that $\{A = 1;\ B = L;\ C = 0;\ D = 1\}$ and the transformation through a thin lens is given by

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f}$$

so that $\left\{A = 1;\ B = 0;\ C = -\frac{1}{f};\ D = 1\right\}$.

Further **"justification"** of this transformation law may be found in terms of the so called "$\rho/\rho'$ argument"-- *viz*




From geometric optics and, in particular, Equation [ II-11 ], we may write

$$\frac{\rho_{\text{out}}}{\rho'_{\text{out}}} = \left( A\,\frac{\rho_{\text{in}}}{\rho'_{\text{in}}} + B \right) \left( C\,\frac{\rho_{\text{in}}}{\rho'_{\text{in}}} + D \right)^{-1}$$

or

$$\Delta z_{\text{out}} = \left( A\,\Delta z_{\text{in}} + B \right) \left( C\,\Delta z_{\text{in}} + D \right)^{-1}$$

which is identical to transformation equation [ III-29 ] if we interpret $q(z)$ as the wave optics generalization of $\Delta z = \rho / \rho'$ .