ON CLASSICAL ELECTROMAGNETIC FIELDS

I. PRELIMINARIES: A REVIEW OF SOME BASIC CONCEPTS AND METHODS:

THE MICROSCOPIC MAXWELL'S EQUATIONS IN THE TIME DOMAIN:

$$\vec{\;}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=-\frac{\partial}{\partial t}\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-1a]}$$

$$\vec{\;}\times\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)=\mu_0\,\vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right)+\varepsilon_0\,\mu_0\frac{\partial}{\partial t}\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-1b]}$$

$$\vec{\;}\;\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=\frac{1}{\varepsilon_0}\rho\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-1c]}$$

$$\vec{\;}\;\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-1d]}$$

where the meaning of fields $\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)$ and $\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)$ is ultimately defined in terms of the Lorentz force on a charge q moving at a velocity $\vec{\mathbf{u}}$ -- *viz.*,

$$\vec{\mathbf{F}}\left(\vec{\mathbf{r}},t\right)=q\,\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)+q\,\vec{\mathbf{u}}\times\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-2]}$$

THE MACROSCOPIC MAXWELL'S EQUATIONS IN THE TIME DOMAIN

Following common practice, we set forth a particular set of macroscopic Maxwell’s equations that applies in the high frequency or ***optical regime***.[1]

$$\vec{\;}\times\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=-\frac{\partial}{\partial t}\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)=-\mu_0\frac{\partial}{\partial t}\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-3a]}$$

$$\vec{\;}\times\vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)=\mu_0\;\vec{\;}\times\vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right)=\mu_0\left[\vec{\mathbf{J}}\left(\vec{\mathbf{r}},t\right)+\frac{\partial}{\partial t}\vec{\mathbf{P}}\left(\vec{\mathbf{r}},t\right)\right]+\varepsilon_0\mu_0\frac{\partial}{\partial t}\vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right) \qquad \text{[I-3b]}$$

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1 All bound current effects are included in the polarization density, since magnetization density "ceases to have any physical meaning at relatively low frequencies." See Section 62 in L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press (1960).

$$\vec{\;}\ \vec{\mathbf{E}}\left(\vec{\mathbf{r}},t\right)=\frac{1}{\varepsilon_0}\left[\rho\left(\vec{\mathbf{r}},t\right)-\vec{\;}\ \vec{\mathbf{P}}\left(\vec{\mathbf{r}},t\right)\right] \qquad \text{[I-3c]}$$

$$\vec{\;}\ \vec{\mathbf{B}}\left(\vec{\mathbf{r}},t\right)=\mu_0\ \vec{\;}\ \vec{\mathbf{H}}\left(\vec{\mathbf{r}},t\right)=0 \qquad \text{[I-3d]}$$

A PHENOMENOLOICAL REPRESENTATION OF THE LINEAR DIELECTRIC RESPONSE OF MATTER:[2]

The following is the most general phenomenological representation of the linear dielectric response of a given material that incorporates **dissipative**, **non-local**, and **anisotropic** effects:

In a ***tensor representation***

$$\mathrm{P}_\alpha^{(\mathrm{L})}\left(\vec{\mathbf{r}},t\right)=\varepsilon_0\int_{-\infty}^{t}dt'\iiint\chi_{\alpha\beta}\left(\vec{\mathbf{r}},\vec{\mathbf{r}}';\ t,t'\right)\ \mathrm{E}_\beta\left(\vec{\mathbf{r}},t'\right)d\vec{\mathbf{r}}' \qquad \text{[I-4a]}$$

In a ***dyadic representation***

$$\vec{\mathbf{P}}^{(\mathrm{L})}\left(\vec{\mathbf{r}},\ t\right)=\varepsilon_0\int_{-\infty}^{t}dt'\iiint\vec{\vec{\chi}}\left(\vec{\mathbf{r}},\vec{\mathbf{r}}';\ t,t'\right)\ \vec{\mathbf{E}}\left(\vec{\mathbf{r}},\ t'\right)d\vec{\mathbf{r}}' \qquad \text{[I-4b]}$$

For the present and for most of our discussions, we neglect **nonlocal** effects and treat only **dispersive** (dissipative) and **anisotropic** effects so that

2 In our later treatment of nonlinear optics, we will begin by adding the following nonlinear phenomenological contributions:

$$\begin{aligned}\mathbf{P}_\alpha^{(\mathrm{NL})}\left(\vec{\mathbf{r}},t\right)=\ &\varepsilon_0\sum_{\beta\gamma}\int_{\vec{\mathbf{r}}_1}\int_{\vec{\mathbf{r}}_2}\int_{t_1}\int_{t_2}d\vec{\mathbf{r}}_1dt_1d\vec{\mathbf{r}}_2dt_2\ \chi_{\alpha\beta\gamma}^{(2)}\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1,\ t-t_1;\vec{\mathbf{r}}-\vec{\mathbf{r}}_2,\ t-t_2\right)\mathbf{E}_\beta\left(\vec{\mathbf{r}}_1,\ t_1\right)\mathbf{E}_\gamma\left(\vec{\mathbf{r}}_2,\ t_2\right)\\ &+\varepsilon_0\sum_{\beta\gamma\delta}\int_{\vec{\mathbf{r}}_1}\int_{\vec{\mathbf{r}}_2}\int_{\vec{\mathbf{r}}_3}\int_{t_1}\int_{t_2}\int_{t_3}d\vec{\mathbf{r}}_1dt_1d\vec{\mathbf{r}}_2dt_2d\vec{\mathbf{r}}_3dt_3\ \chi_{\alpha\beta\gamma\delta}^{(3)}\left(\vec{\mathbf{r}}-\vec{\mathbf{r}}_1,\ t-t_1;\vec{\mathbf{r}}-\vec{\mathbf{r}}_2,\ t-t_2;\vec{\mathbf{r}}-\vec{\mathbf{r}}_3,\ t-t_3\right)\\ &\times\mathbf{E}_\beta\left(\vec{\mathbf{r}}_1,\ t_1\right)\mathbf{E}_\gamma\left(\vec{\mathbf{r}}_2,\ t_2\right)\mathbf{E}_\delta\left(\vec{\mathbf{r}}_3,\ t_3\right)+\cdots.\end{aligned}$$

$$\vec{\mathbf{P}}\left(\vec{\mathbf{r}}, t\right) = \varepsilon_0 \int_{-\infty}^{t} dt' \; \overleftrightarrow{\chi}\left(\vec{\mathbf{r}}, \, t - t'\right) \; \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, t'\right) \qquad [\text{ I-5 }]$$

or

$$\vec{\mathbf{P}}\left(\vec{\mathbf{r}}, \omega\right) = \varepsilon_0 \, \overleftrightarrow{\chi}\left(\vec{\mathbf{r}}, \omega\right) \; \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, \omega\right) \qquad [\text{ I-6 }]$$

where

$$\overleftrightarrow{\chi}\left(\vec{\mathbf{r}}, \omega\right) = \int_{-\infty}^{t} \overleftrightarrow{\chi}\left(\vec{\mathbf{r}}, t'\right) \exp\left[-i\,\omega\left(t - t'\right)\right] dt' = \int_{0}^{\infty} \overleftrightarrow{\chi}\left(\vec{\mathbf{r}}, \tau\right) \exp\left[-i\omega\tau\right] d\tau \qquad [\text{ I-7 }]$$

Macroscopic Maxwell's Equations in the Frequency Domain Valid for Linear, Local, Anisotropic Media in the *Optical Regime*.

$$\vec{\nabla} \times \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, \omega\right) = \vec{\nabla} \times \left[\overleftrightarrow{\varepsilon}\right]^{-1}\left(\vec{\mathbf{r}}, \omega\right) \; \vec{\mathbf{D}}\left(\vec{\mathbf{r}}, \omega\right) = -i\omega \vec{\mathbf{B}}\left(\vec{\mathbf{r}}, \omega\right) = -i\omega\mu_0 \vec{\mathbf{H}}\left(\vec{\mathbf{r}}, \omega\right) \qquad [\text{ I-8a }]$$

$$\begin{aligned} \vec{\nabla} \times \vec{\mathbf{B}}\left(\vec{\mathbf{r}}, \omega\right) &= \mu_0 \vec{\nabla} \times \vec{\mathbf{H}}\left(\vec{\mathbf{r}}, \omega\right) = \mu_0 \vec{\mathbf{J}}\left(\vec{\mathbf{r}}, \omega\right) + i\omega\mu_0 \overleftrightarrow{\varepsilon}\left(\vec{\mathbf{r}}, \omega\right) \; \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, \omega\right) \\ &= \mu_0 \vec{\mathbf{J}}\left(\vec{\mathbf{r}}, \omega\right) + i\omega\mu_0 \vec{\mathbf{D}}\left(\vec{\mathbf{r}}, \omega\right) \end{aligned} \qquad [\text{ I-8b }]$$

$$\vec{\nabla} \cdot \overleftrightarrow{\varepsilon}\left(\vec{\mathbf{r}}, \omega\right) \; \vec{\mathbf{E}}\left(\vec{\mathbf{r}}, \omega\right) = \vec{\nabla} \cdot \vec{\mathbf{D}}\left(\vec{\mathbf{r}}, \omega\right) = \rho\left(\vec{\mathbf{r}}, \omega\right) \qquad [\text{ I-8c }]$$

$$\vec{\nabla} \cdot \vec{\mathbf{B}}\left(\vec{\mathbf{r}}, \omega\right) = \mu_0 \vec{\nabla} \cdot \vec{\mathbf{H}}\left(\vec{\mathbf{r}}, \omega\right) = 0 \qquad [\text{ I-8d }]$$

HELMHOLZ EQUATIONS FOR THE FREQUENCY DOMAIN VECTOR AND SCALAR POTENTIALS IN A UNIFORM, LINEAR, ISOTROPIC DIELECTRIC

We define the (Magnetic) Vector Potential as

$$\vec{\mathbf{H}}\left(\vec{\mathbf{r}},\omega\right)=\frac{1}{\mu_0}\,\vec{\mathbf{B}}\left(\vec{\mathbf{r}},\omega\right)=\frac{1}{\mu_0}\,\vec{\nabla}\times\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-9]}$$

which, by design, automatically satisfies one of Maxwell's equations -- *viz.*, [I-8d].[3] . We introduce the (Electric) Scalar Potential in the form

$$\vec{\mathbf{E}}\left(\vec{\mathbf{r}},\omega\right)=-i\omega\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)-\vec{\nabla}\varphi\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-10]}$$

which, again, automatically satisfies another Maxwell equation -- *viz.*, [I-8a].[4] Therefore, for **uniform, isotropic media** Equation [I-8b] becomes [5]

$$\begin{aligned}\vec{\nabla}\times\vec{\nabla}\times\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)&=\mu_0\,\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)+\omega^2\mu_0\varepsilon(\omega)\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)-i\omega\mu_0\varepsilon(\omega)\,\vec{\nabla}\varphi\left(\vec{\mathbf{r}},\omega\right)\\&=\vec{\nabla}\vec{\nabla}\cdot\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)-\nabla^2\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)\end{aligned} \qquad \text{[I-11a]}$$

and Equation [I-8c] becomes

$$-i\omega\,\varepsilon(\omega)\,\vec{\nabla}\cdot\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)-\varepsilon(\omega)\,\nabla^2\varphi\left(\vec{\mathbf{r}},\omega\right)=\rho\left(\vec{\mathbf{r}},\omega\right). \qquad \text{[I-11b]}$$

Since $\vec{\nabla}\cdot\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)$ is as yet undefined, we define it in the ***Lorentz gauge*** as

$$\vec{\nabla}\cdot\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)=-i\omega\mu_0\,\varepsilon(\omega)\,\varphi\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-12]}$$

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3 Since $\quad div\ curl\left\{\quad\right\}=\vec{\nabla}\cdot\vec{\nabla}\times\left\{\quad\right\}=0.$

4 Since $\quad curl\ grad\left\{\quad\right\}=\vec{\nabla}\times\vec{\nabla}\left\{\quad\right\}=0.$

5 Using $\quad \vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right)=\vec{\mathbf{b}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right)-\vec{\mathbf{c}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right).$

to simplify Equations [I-11a] and [I-11b]. Thus

$$^{2}\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)+\omega^{2}\mu_{0}\,\varepsilon\left(\omega\right)\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)=-\mu_{0}\,\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-13a]}$$

and

$$^{2}\varphi\left(\vec{\mathbf{r}},\omega\right)+\omega^{2}\mu_{0}\,\varepsilon\left(\omega\right)\varphi\left(\vec{\mathbf{r}},\omega\right)=-\frac{1}{\varepsilon\left(\omega\right)}\rho\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-13b]}$$

Therefore, in the Lorentz gauge **both** $\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)$ and $\varphi\left(\vec{\mathbf{r}},\omega\right)$ **satisfy inhomogeneous (**and homogeneous) **Helmholz equations!**

However, in the ***Coulomb gauge*** we define $\vec{\ }\ \vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)=0$ and then Equation [I-8c] becomes

$$^{2}\varphi\left(\vec{\mathbf{r}},\omega\right)=-\frac{1}{\varepsilon\left(\omega\right)}\rho\left(\vec{\mathbf{r}},\omega\right). \qquad \text{[I-14a]}$$

Conservation of charge requires that $\vec{\ }\ \vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)-i\omega\rho\left(\vec{\mathbf{r}},\omega\right)=0$ so that Equation [I-11a] becomes

$$^{2}\,\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)+\omega^{2}\mu_{0}\,\varepsilon\left(\omega\right)\vec{\mathbf{A}}\left(\vec{\mathbf{r}},\omega\right)=-\mu_{0}\,\vec{\mathbf{J}}^{(\mathrm{trans})}\left(\vec{\mathbf{r}},\omega\right) \qquad \text{[I-14b]}$$

where

$$\begin{aligned}\vec{\mathbf{J}}^{(\mathrm{trans})}\left(\vec{\mathbf{r}},\omega\right)&=\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)-\vec{\mathbf{J}}^{(\mathrm{long})}\left(\vec{\mathbf{r}},\omega\right)\\&=\vec{\mathbf{J}}\left(\vec{\mathbf{r}},\omega\right)-\left[-i\omega\varepsilon\left(\omega\right)\quad\varphi\left(\vec{\mathbf{r}},\omega\right)\right]\end{aligned} \qquad \text{[I-15]}$$