Lipschitz Conditions

We define the function $f(x)$ to satisfy a Lipschitz condition on the interval $[a, b]$ if there exists a constant K (dependent on both f and the interval) such that

$$|f(x_1) - f(x_2)| < K|x_1 - x_2|.$$

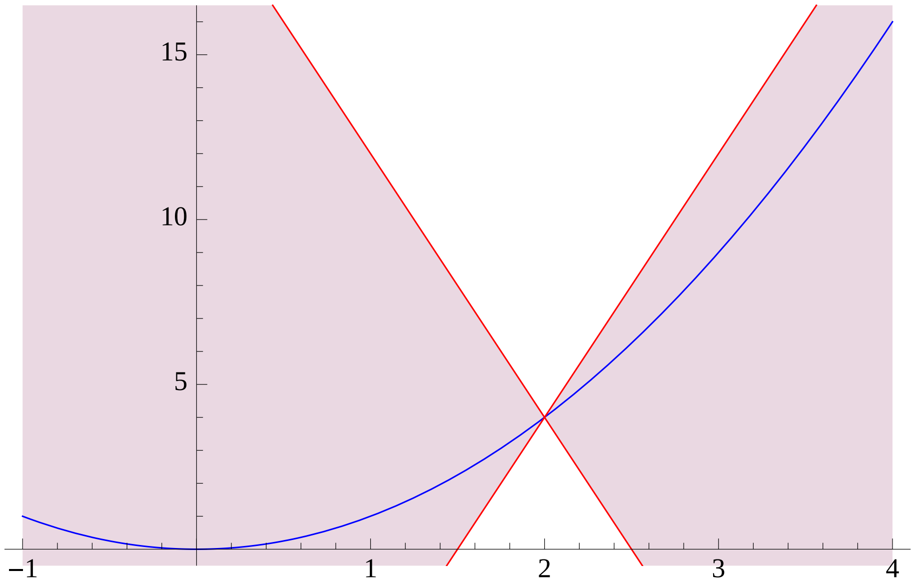
(Note, in general, K will also depend on the choice of norm but for equivalent norms there will be a Lipschitz condition with respect to one if and only if there is one with respect to the other.)

Example 1 $f(x) = x^2$ on $[-1, 4]$.

$$|f(x_1) - f(x_2)| = |{x_1}^2 - {x_2}^2| = |(x_1 + x_2)(x_1 - x_2)| \leq \max x_1, x_2 \in [-1, 4]|x_1 + x_2||x_1 - x_2| = 8|x_1 - x_2|$$

i.e. we can take $K = 8$ in this case.

Graphical interpretation: for any point on the curve all other points lie within the region defined by the lines with slope ± 8 through that point:



Generally if $f(x)$ is differentiable on $[a, b]$ then the Mean Value Theorem says that

$$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2).$$

for some ξ between x_1 and x_2. and as a result

$$|f(x_1) - f(x_2)| = |f'(\xi)|\cdot|x_1 - x_2| = \max_{\xi \in [a,b]} |f'(\xi)|\cdot|x_1 - x_2|,$$

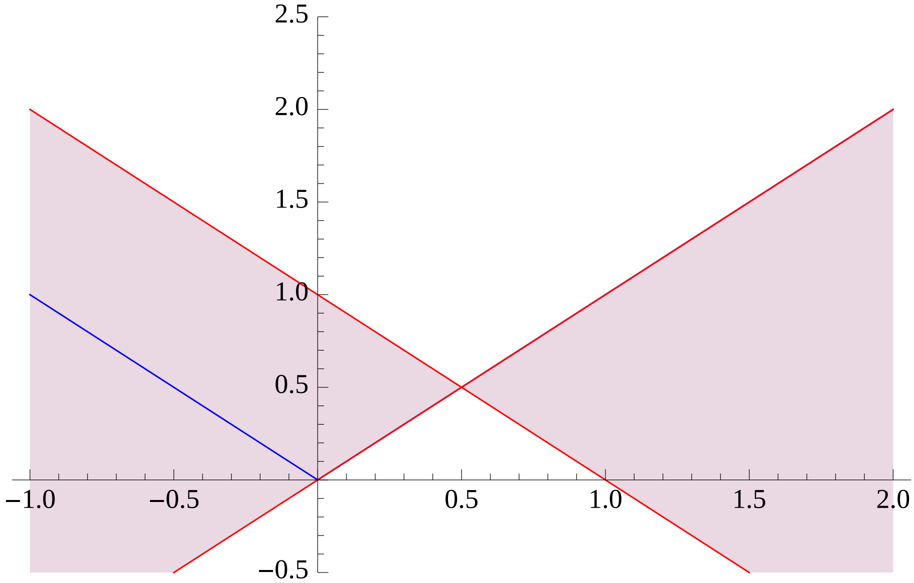
assuming that $|f'(\xi)|$ is bounded in $[a, b]$ which will certainly be the case if $f'(x)$ is continuous, ie, if f is continuously differentiable.

Example 1 (revisited) $f(x) = x^2 \implies f'(x) = 2x$ so we can take $K = \max_{\xi \in [-1,4]} |f'(\xi)| = \max_{\xi \in [-1,4]} 2|\xi| = 8$, as before.

Example 2 $f(x) = |x|$ on $[a, b]$

By the above we'll have a Lipschitz condition on any bounded interval on which $f(x)$ is continuously differentiable, ie any interval not containing 0 (with Lipschitz constant $|f'(\xi)| = 1$,. So consider an interval containing 0, eg $[-1, 2]$ and let $x_1 \le 0$, $x_2 \ge 0$ then

$$f(x_1)-f(x_2) = -x_1-x_2 < -x_1+x_2 = |x_1-x_2| \qquad \text{and} \qquad f(x_1)-f(x_2) = -x_1-x_2 > x_1-x_2 = -|x_1-x_2|.$$



Example 3 $f(x) = |x|^{1/2}$ on $[a, b]$

By the above we'll have a Lipschitz condition on any bounded interval on which $f(x)$ is continuously differentiable, ie any interval not containing 0. In this case though we might be concerned since of $\epsilon > 0$ and we consider the interval $[\epsilon, 1]$ our argument above gave Lipschitz constant

$$K_\epsilon = \max_{\xi \in [\epsilon,1]} |f'(\xi)| = \max_{\xi \in [\epsilon,1]} (1/2)|\xi|^{-1/2} = (1/2)|\epsilon^{-1/2}$$

and clearly $K_\epsilon \to \infty$ as $\epsilon \to 0$.

To see that no K could work take $x_2 = 0$ then we would need

$$|x_1|^{1/2} < K|x_1| \equiv K > |x_1|^{-1/2}$$

for all x_1 in our interval including 0, but this is clearly impossible as $|x_1|^{-1/2} \to \infty$ as $x_1 \to 0$.

Graphically our limiting lines have to get steeper and steeper as we approach 0.