

3 Lipschitz condition and Lipschitz continuity

3.1 Definition (Lipschitz-continuous function) Lipschitz-continuity comes in three different flavours. Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$.

(a) Given an open set $B \subseteq \mathbb{R}^m$, we say that f is *Lipschitz-continuous on the open subset B* if there exists a constant $\Lambda \in \mathbb{R}_0^+$ (called the *Lipschitz constant of f on B*) such that

$$\|f(x) - f(y)\| \leq \Lambda \|x - y\|, \quad \forall x, y \in B. \quad (15)$$

(b) The function f is called *locally Lipschitz-continuous*, if for each $z \in \mathbb{R}^n$ there exists an $L > 0$ such that f is Lipschitz-continuous on the open ball of center z and radius L

$$B_L(z) := \{y \in \mathbb{R}^m : \|y - z\| < L\}. \quad (16)$$

(c) If f is Lipschitz continuous on all of the space \mathbb{R}^m (i.e. $B = \mathbb{R}^m$ in (15)), then f is called *globally Lipschitz-continuous*.

3.2 Remark (local vs. global) Notice the fundamental difference between the local and global versions of the Lipschitz-continuity. Whereas in the local version the Lipschitz constant Λ and the open set B depend on each point $z \in \mathbb{R}^m$, in the global version the constant Λ is fixed and $B = \mathbb{R}^m$. In particular, a *globally Lipschitz-continuous function* is *locally Lipschitz-continuous*, but the viceversa is not true.

3.3 Remark (norms and Lipschitz constants) In (15) the norm $\|\cdot\|$ can be any norm. However, once a norm has been chosen one should stick to that single norm, as the Lipschitz constant Λ depends on the particular choice of this norm. Unless otherwise stated, we use the Euclidean norm in all our analysis.

3.4 Interpretation

To see what these definitions mean, let us consider the situation in one dimension. Suppose f is a Lipschitz function on a neighborhood B of $x \in \mathbb{R}$. This implies that, $\forall y \in B$,

$$\begin{aligned} |f(x) - f(y)| \leq \Lambda |x - y| &\Rightarrow \left| \frac{f(x) - f(y)}{x - y} \right| \leq \Lambda \\ \Rightarrow \left| \frac{f(x+h) - f(x)}{h} \right| \leq \Lambda &\text{ by putting } y = x + h. \end{aligned}$$

If we were to let $h \rightarrow 0$ and if the function f were differentiable, then the result would mean $|f'(x)| \leq \Lambda$; that is that the derivative is bounded by the Lipschitz constant.

However, there is nothing in the definition of Lipschitz-continuity that implies that f is differentiable. So in general we can't proceed to this limit, since we don't know if f is differentiable at x . But this tells us all we need to know: being Lipschitz just means f can't be too steep, the bound on the difference quotient being Λ .

3.5 Examples and Counterexamples

The functions below are pictured in figure 1; we will examine them in turn, with $B = [-1, 1]$.

3.6 Example Consider $f_1(x) = x^2$. We will show that f_1 is locally Lipschitz-continuous but not globally so.

This function is continuously differentiable. Pick up any point $x \in \mathbb{R}$, we observe that

$$\sup_{y \in (x-1, x+1)} |f'_1(y)| = \sup_{y \in (x-1, x+1)} |2y| \leq 2|x| + 1 \quad (17)$$

being $|y| \leq |x| + 1$ for $y \in (x-1, x+1)$ by the triangle inequality. Now picking up two points, $y, z \in (x-1, x+1)$ it follows by the mean value theorem that for some ξ between y and z , that

$$|f_1(z) - f_1(y)| = |f'_1(\xi)(z - y)| \leq \sup_{\theta \in (x-1, x+1)} |f'_1(\theta)| |z - y|. \quad (18)$$

Using (17), we conclude that

$$|f_1(z) - f_1(y)| \leq (2|x| + 1) |z - y|, \quad \forall z, y \in (x-1, x+1). \quad (19)$$

Hence the Lipschitz constant of f_1 on $(x-1, x+1)$ is $\Lambda = 2|x| + 1$ and the function is locally Lipschitz-continuous on all of \mathbb{R} .

Observe that the Lipschitz constant depends on x and its neighbourhood. In particular, if $x \rightarrow \infty$ then $\Lambda \rightarrow \infty$. This is an indication that the function may not be globally Lipschitz continuous. Indeed, for any $y \neq 0$

$$\frac{|f_1(y) - f_1(0)|}{|y - 0|} = |y| \rightarrow \infty, \quad \text{as } y \rightarrow \infty, \quad (20)$$

which means that there is no Λ that can satisfy the global Lipschitz property for $y \in \mathbb{R}$.

3.7 Example $f_2(x) = \begin{cases} x^2 \sin(1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ This function is differentiable everywhere:

$$f'_2(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(One has to use the definition of derivative to obtain $f'_2(0)$.)

However, since $f'_2(x) \rightarrow 0$ as $x \rightarrow 0$, the derivative is not continuous. Is f_2 Lipschitz-continuous?

Define $x_n = (2n\pi + \pi/2)^{-1/2}$ and $y_n = (2n\pi)^{-1/2}$ for $n = 1, 2, \dots$, and suppose that f_2 is Lipschitz-continuous. Since $x_n, y_n \in B$ for all n , there must exist Λ such that

$$\begin{aligned} \Lambda &\geq \left| \frac{f_2(x_n) - f_2(y_n)}{x_n - y_n} \right| \quad \text{for all } n \\ &= \frac{(2n\pi + \frac{\pi}{2})^{-1}}{(2n\pi)^{-1/2} - (2n\pi + \frac{\pi}{2})^{-1/2}} \\ &= 4n \left((2n\pi)^{-1/2} + \left(2n\pi + \frac{\pi}{2}\right)^{-1/2} \right) \rightarrow \infty \text{ as } n \rightarrow \infty. \end{aligned}$$

Plainly no such Λ can exist. Hence f_2 is not Lipschitz-continuous.

In the first example we used the theorem “continuously differentiable \implies locally Lipschitz”. Here we saw an example where a differentiable function whose derivative is not continuous turned out not to be Lipschitz. However, this is not true of all such functions. For example, consider the function (shown in the handout ‘2.2 Continuity and Differentiability—some facts of life’)

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is differentiable, but f' is not continuous. It is left as an exercise to show that f is globally Lipschitz continuous.

3.8 Example This example shows that differentiability is a stronger concept than Lipschitz continuity. $f_3(x) = |x|$ This function is not differentiable at $x = 0$, but

$$f'_3(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Since

$$|f_3(x) - f_3(y)| = ||x| - |y|| \leq |x - y|, \quad \text{for all } x, y \in \mathbb{R},$$

we see that f_3 is globally Lipschitz continuous, with Lipschitz constant $\Lambda = 1$.

3.9 Example $f_4(x) = |x|^{1/2}$ This function is not differentiable at $x = 0$, but

$$f'_4(x) = \begin{cases} -\frac{1}{2}(-x)^{-1/2} & \text{if } x < 0, \\ \frac{1}{2}x^{-1/2} & \text{if } x > 0. \end{cases}$$

Is f_4 Lipschitz-continuous?

Define $x_n = 1/n^2$ and $y_n = 0$ for $n = 1, 2, \dots$, and suppose that f_4 is Lipschitz-continuous. Since $x_n, y_n \in B$ for all n , there must exist Λ such that

$$\begin{aligned} \Lambda &\geq \left| \frac{f_4(x_n) - f_4(y_n)}{x_n - y_n} \right| && \text{for all } n \\ &= \frac{1}{n} && \text{as } n \rightarrow \infty. \end{aligned}$$

Plainly no such Λ can exist. Thus f_4 is not Lipschitz-continuous.

3.10 Relationships between Lipschitz continuity, continuity and differentiability

3.11 Theorem (Lipschitz $\implies C^0$) *Every locally Lipschitz-continuous function is continuous.*

The proof of this Theorem is left as an exercise.

3.12 Theorem ($C^1 \implies$ Lipschitz) *Every continuously differentiable function is locally Lipschitz.*

Proof Let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be continuously differentiable. Fix any two points $\mathbf{z}, \mathbf{y} \in \mathbb{R}^m$ and define the function $f : [0, 1] \rightarrow \mathbb{R}$ as $f(\theta) := \mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z}))$. It is clear that

$$f(0) = \mathbf{f}(\mathbf{z}) \text{ and } f(1) = \mathbf{f}(\mathbf{y}). \quad (21)$$

Furthermore, by the chain rule, we know that f is differentiable and that

$$\frac{d}{d\theta} f(\theta) = D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z})) \cdot (\mathbf{y} - \mathbf{z}). \quad (22)$$

Here $D\mathbf{f}(\mathbf{w})$ is the Jacobian matrix of \mathbf{f} at \mathbf{w} :

$$D\mathbf{f}(\mathbf{w}) = \begin{pmatrix} \frac{\partial f_1}{\partial w_1}(\mathbf{w}) & \cdots & \frac{\partial f_1}{\partial w_m}(\mathbf{w}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial w_1}(\mathbf{w}) & \cdots & \frac{\partial f_m}{\partial w_m}(\mathbf{w}) \end{pmatrix} \quad (23)$$

where $\frac{\partial f_i}{\partial w_j}(\mathbf{w})$ is the partial derivative of the i -th component of f with respect to the j -th coordinate.

By the definition of f and then by the fundamental theorem of calculus, we have

$$\begin{aligned} \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z}) &= f(1) - f(0) = \int_0^1 f'(\theta) d\theta \\ &= \left(\int_0^1 D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z})) d\theta \right) (\mathbf{y} - \mathbf{z}); \end{aligned}$$

where the integral in the last line is a matrix whose i, j -th component is given by

$$\int_0^1 \frac{\partial f_i}{\partial w_j}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z})) d\theta.$$

It follows that

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z})\| \leq \left\| \int_0^1 D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z})) d\theta \right\| \|\mathbf{y} - \mathbf{z}\|,$$

where we use the notation $\|\cdot\|$ for both the vector norm and its associated matrix norm.

Notice that, by the triangle inequality for integrals, we have

$$\begin{aligned} \left\| \int_0^1 D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z})) d\theta \right\| &\leq \int_0^1 \|D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z}))\| d\theta \\ &\leq \sup_{\theta \in [0,1]} \|D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z}))\| \int_0^1 d\theta \\ &= \sup_{\theta \in [0,1]} \|D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z}))\|. \end{aligned}$$

Furthermore by the equivalence of matrix norms we have

$$\exists c_0 > 0 : \|A\| \leq c_0 \|A\|_\infty, \forall A \in \mathbb{R}^{m \times m}, \quad (24)$$

where c_0 depends only on m and $\|\cdot\|_\infty$ is the maximum row sum norm from last handout.

Thus, to establish Lipschitz continuity, we fix an arbitrary point $\mathbf{x} \in \mathbb{R}^m$ and we establish a bound for $\int_0^1 D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z})) d\theta$ in an appropriate neighbourhood B of \mathbf{x} .

Let $B = B_L(\mathbf{x})$, with L arbitrary. Since \mathbf{f} is continuously differentiable on \overline{B} , which is closed and bounded, there exists Λ_0 such that

$$\sup_{\mathbf{w} \in B} \left\| \frac{\partial f_i}{\partial w_j}(\mathbf{w}) \right\| \leq \Lambda_0, \forall i, j \in [1 : m].$$

Here we have applied the Weierstrass theorem which says that each continuous function $(\partial_{w_j} f_i)$ is bounded on a closed and bounded set (B) .

Now, given $\mathbf{z}, \mathbf{y} \in B$, it follows that $\mathbf{z} + \theta(\mathbf{y} - \mathbf{z}) \in B$ for all $\theta \in [0, 1]$ because B is a ball, so

$$\sup_{\theta \in [0, 1]} \|D\mathbf{f}(\mathbf{z} + \theta(\mathbf{y} - \mathbf{z}))\| \leq c_0 m \Lambda_0 =: \Lambda.$$

It follows that

$$\|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{z})\| \leq \Lambda \|\mathbf{y} - \mathbf{z}\|, \forall \mathbf{y}, \mathbf{z} \in B;$$

which is to say that \mathbf{f} is Lipschitz-continuous on $B_L(\mathbf{x})$. Since \mathbf{x} is arbitrary this means that \mathbf{f} is locally Lipschitz-continuous on \mathbb{R}^m . ■

3.13 Remark Notice that in the proof above Λ_0 , and therefore Λ , might depend on the point \mathbf{x} and the constant L . However, if we can find a bound that is independent of \mathbf{x} and L (i.e., a *uniform* bound), then it means that the function \mathbf{f} is actually globally Lipschitz-continuous.

3.14 Corollary If $\|D\mathbf{f}(w)\|$ is uniformly bounded as $\|w\| \rightarrow \infty$, then \mathbf{f} is globally Lipschitz.

3.15 Lipschitz-continuity with respect to some arguments

As you may have noticed in the course notes we used a slightly different version of Lipschitz continuity: the *Lipschitz continuity with respect to the first argument* of a function

$$\mathbf{f} : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}. \quad (25)$$

We give now an adaptation of our earlier definition. The main point is that the only variable that counts is the first one. It is useful though to have the function continuous with respect to both variables.

3.16 Definition A function \mathbf{f} such as in (25) is *locally Lipschitz continuous with respect to its first argument* if it is continuous and if for each $(\mathbf{x}, t) \in \mathbb{R}^m \times \mathbb{R}$ there exists a $L > 0$ and a $\Lambda > 0$ such that

$$\|\mathbf{f}(\mathbf{y}, s) - \mathbf{f}(\mathbf{z}, s)\| \leq \Lambda \|\mathbf{z} - \mathbf{y}\| \quad (26)$$

for all $\mathbf{z}, \mathbf{y} \in B_L(\mathbf{x})$ and $s \in (t - L, t + L)$. (Notice how the right-hand side is independent of the second argument s .)

The function \mathbf{f} is called *globally Lipschitz-continuous with respect to its first argument* if it is continuous and if there is a $\Lambda > 0$ such that (26) is satisfied for all \mathbf{z}, \mathbf{y} in \mathbb{R}^m and $s \in \mathbb{R}$. (Notice again the secondary role played by the time variable.)

A fact that we have used in the proof of uniqueness is the following:

3.17 Lemma (Characterization of local Lipschitz-continuity) *A continuous function \mathbf{f} , such as in (25), is locally Lipschitz continuous with respect to its first argument if and only if for each closed and bounded subset K of $\mathbb{R}^m \times \mathbb{R}$, there exists an open set $A \subset \mathbb{R}^m$ such that $K \subset A$ and a constant $\Lambda_K > 0$ such that*

$$\|\mathbf{f}(\mathbf{x}, t) - \mathbf{f}(\mathbf{y}, t)\| \leq \Lambda_K \|\mathbf{x} - \mathbf{y}\|, \quad \forall (\mathbf{x}, t), (\mathbf{y}, t) \in A. \quad (27)$$

Proof (For the bold and the knowledgeable.) The “if” part is easy and is left as an exercise. The “only if” part can be proved by contradiction. Suppose that the conclusion is false. This means is, that there exists a closed and bounded subset $K_0 \subset \mathbb{R}^m \times \mathbb{R}$ such that any $n \in \mathbb{Z}^+$ there exist $\mathbf{x}_n, \mathbf{y}_n \in \mathbb{R}^m$, $t_n \in \mathbb{R}$, such that $(\mathbf{x}_n, t_n), (\mathbf{y}_n, t_n) \in A_n$, with $A_n = \{(\mathbf{z}, s) \in \mathbb{R}^m \times \mathbb{R} : \exists (\mathbf{y}, t) \in K_0 : \|\mathbf{z} - \mathbf{y}\| + |s - t| \leq 1/n\}$ and for which

$$\|\mathbf{f}(\mathbf{x}_n, t_n) - \mathbf{f}(\mathbf{y}_n, t_n)\| > n \|\mathbf{x}_n - \mathbf{y}_n\|. \quad (28)$$

Notice that $A_n \subset \bar{A}_1$ which is closed and bounded. It follows that the sequences $((\mathbf{x}_n, t_n))_{n \in \mathbb{Z}^+}$ and $((\mathbf{y}_n, t_n))_{n \in \mathbb{Z}^+}$ are bounded. There exists thus a family of integers $X \subseteq \mathbb{Z}^+$ and a point $(\mathbf{x}^*, t^*) \in \bar{A}$ (the closure of A) such that

$$\lim_{n \rightarrow \infty} (\mathbf{x}_n, t_n) = (\mathbf{x}^*, t^*), n \in X. \quad (29)$$

Likewise, there exists $(\mathbf{y}^*, t^*) \in \bar{A}$ and $Y \subseteq X$ such that

$$\lim_{n \rightarrow \infty} (\mathbf{y}_n, t_n) = (\mathbf{y}^*, t^*), n \in Y. \quad (30)$$

Notice that $(\mathbf{x}^*, t^*), (\mathbf{y}^*, t^*) \in \bigcap_{n \in Y} A_n = K_0$. We show next that $\mathbf{x}^* = \mathbf{y}^*$. Indeed, by the continuity of \mathbf{f} , it has to be bounded by a constant M over the closed and bounded set K_0 . So (28) implies that $\|\mathbf{x}_n - \mathbf{y}_n\| \leq 2M/n$, for $n \in Y$, which means, by passing to the limit, as $n \rightarrow \infty$, that

$$\|\mathbf{x}^* - \mathbf{y}^*\| = 0, \text{ i.e., } \mathbf{x}^* = \mathbf{y}^*. \quad (31)$$

To conclude the proof it is sufficient to observe that the function \mathbf{f} fails to be Lipschitz-continuous in any neighbourhood of (\mathbf{x}^*, t^*) . Indeed for any $\Lambda > 0$ and for any open ball $B \ni (\mathbf{x}^*, t^*)$, for $n \in Y$ big enough, we have that $(\mathbf{x}_n, t_n), (\mathbf{y}_n, t_n) \in B$, and

$$\|\mathbf{f}(\mathbf{x}_n, t_n) - \mathbf{f}(\mathbf{y}_n, t_n)\| > n \|\mathbf{x}_n - \mathbf{y}_n\| \geq \Lambda \|\mathbf{x}_n - \mathbf{y}_n\|, \quad (32)$$

in contrast with the local Lipschitz-continuity with respect to the first argument. ■

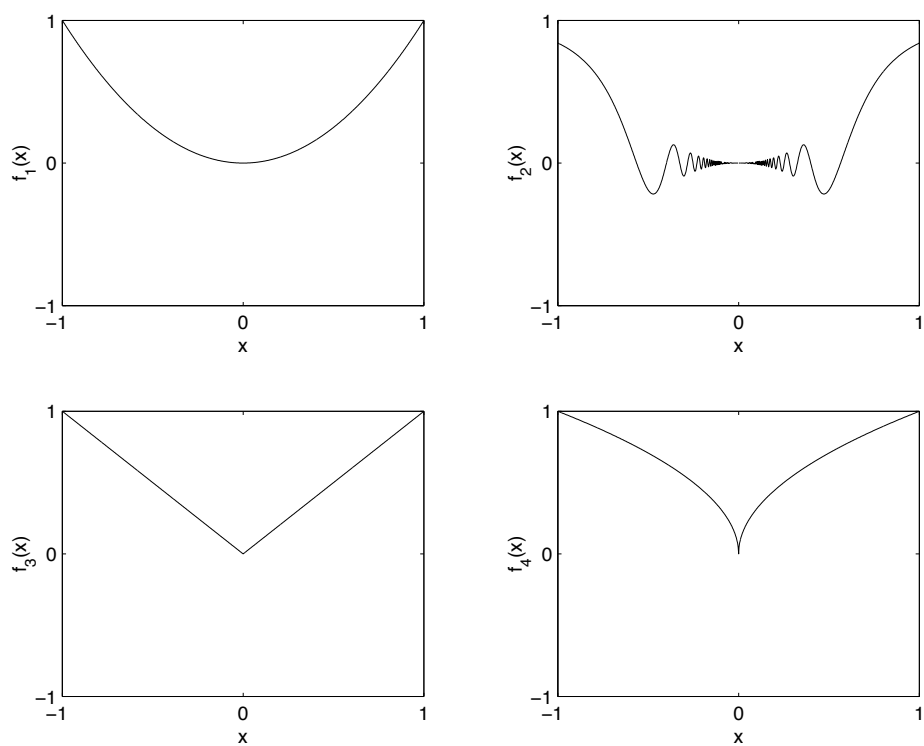


Figure 1: Example functions