Existence and Uniqueness

1 Lipschitz Conditions

We now turn our attention to the general initial value problem

$$\frac{dy}{dt} = f(t, y)$$
$$y(t_0) = y_0,$$

where f is a differentiable function. We would like to know when we have existence of a unique solution for given initial data. One condition on f which guarantees this is the following. Given a subset S of the (t, y)-plane, we say that f is **Lipschitz** with respect to y on the domain S if there is some constant K such that

$$|f(t, y_2) - f(t, y_1)| \leq K|y_2 - y_1| \tag{1}$$

for every pair of points (t, y_1) and (t, y_2) in S. The constant K is called the Lipschitz constant for f on the domain S.

Example 1.1. Let $f(t, y) = 3y + 2$. Then $|f(t, y_2) - f(t, y_1)| = 3|y_2 - y_1|$ so f is Lipschitz with constant 3.

$\diamond$

Example 1.2. Let $f(t, y) = ty^2$. Then since $|f(t, y_2) - f(t, y_1)| = t|y_2 + y_1||y_2 - y_1|$ is not bounded by any constant times $|y_2 - y_1|$, f is not Lipschitz with respect to y on the domain $\mathbb{R} \times \mathbb{R}$. However f is Lipschitz on any rectangle $R = [a, b] \times [c, d]$ since we have $t|y_1 + y_2| \leq 2\max\{|a|, |b|\} \cdot \max\{|c|, |d|\}$ on R.

$\diamond$

The following lemma gives a simple test for a function to be Lipschitz with respect to y.

Lemma 1.1. *Suppose f is continuously differentiable with respect to y on some closed rectangle R. Then f is Lipschitz with respect to y on R.*

Proof. Since $\partial f/\partial y$ is continuous on the closed and bounded set R, it attains a maximum and a minimum on R. Therefore

$$K = \max_{(t,y)\in R_0} \left|\frac{\partial f}{\partial y}(t, y)\right| < \infty.$$

So given (t, y_1) and (t, y_2) in B_0, the Mean Value Theorem implies that there is some y_3 between y_1 and y_2 such that

$$|f(t, y_2) - f(t, y_1)| = \left|\frac{\partial f}{\partial y}(t, y_3)\right| |y_2 - y_1| \leq K|y_2 - y_1|.$$

$\square$

Example 1.3. Let $f(t,y) = |t|e^{ty} + t\sin(t+2y)$. Then $\frac{\partial f}{\partial y} = t|t|e^{ty} + \sin(t+y) - 2t\cos(t+2y)$, which is continuous on any rectangle R. Therefore f is Lipschitz on any rectangle R.

$\diamond$

Note that the converse of Lemma 1.1 is false. That is, Lipschitz with respect to y does not imply differentiable with respect to y.

Example 1.4. Let $f(t,y) = t|y|$ on $R = [-2,2] \times [-2,2]$. Then since

$$|f(t,y_2) - f(t,y_1)| = |t|||y_2| - |y_1|| \leq |t||y_2 - y_1|$$

f is Lipschitz with respect to y on R, with Lipschitz constant $L = 2$. However, $\frac{\partial f}{\partial y}$ is not continuous on R.

$\diamond$

We now state the main theorem about existence and uniqueness of solutions.

Theorem 1.1. *Suppose $f(t,y)$ is continuous in t and Lipschitz with respect to y on the domain $R = [a,b] \times [c,d]$. Then, given any point (t_0, y_0) in R, there exist $\epsilon > 0$ and a unique solution $y(t)$ of the initial value problem*

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0$$

on the interval $(t_0 - \epsilon, t_0 + \epsilon)$.

Note that Theorem 1.1 asserts only the existence of a solution on some interval, which could be quite small in general.

Example 1.5. Consider the equation $\frac{dy}{dt} = y^2$. Since $f(t,y) = y^2$ is Lipschitz on any rectangle in the (t,y)-plane, by Theorem 1.1 for any initial data $y(t_0) = y_0 \in \mathbb{R}$ there is a unique solution on some interval. In this case we can find the solutions explicitly. We can rewrite the equation

$$\frac{dy/dt}{y(t)^2} = 1 \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\frac{-1}{y(t)}\right) = 1.$$

Integrating from 0 to t gives

$$-\frac{1}{y(t)} + \frac{1}{y_0} = t \qquad \Longrightarrow \qquad y(t) = \frac{y_0}{1 - y_0 t}.$$

Now suppose $y_0 > 0$. Then the solution blows up to infinity as t approaches $1/y_0$. Hence the interval containing 0 on which the solution exists is $(-\infty, 1/y_0)$. For large y_0, the interval of positive time for which the solution exists is very small.

$\diamond$

Example 1.6. Consider $\frac{dy}{dt} = y^{2/3}$. Solving this separable equation gives

$$y(t) = \left(\frac{t}{3} + y_0^{1/3}\right)^3.$$

For $y_0 = 0$ we therefore have the solution $y(t) = t^3/27$. However $y(t) \equiv 0$ is also a solution with initial data $y_0 = 0$, so we have non-uniqueness of solutions for this equation. The problem of course is that $f(y) = y^{1/3}$ is not Lipschitz. There is no Lipschitz constant in any interval containing zero since

$$\frac{|f(t,y) - f(t,0)|}{|y-0|} = \frac{1}{|y^{2/3}|} \to \infty \text{ as } y \to 0.$$

Note however that $y_0 = 0$ is the only initial data for which we have non-uniqueness. For if $y_0 > 0$ (the same reasoning applies for $y_0 < 0$), then on the interval $J = (y_0/2, \infty)$ the derivative of f with respect to y is bounded. For $\partial f/\partial y = \frac{2}{3}y^{-1/3}$ is decreasing in y on J and thus

$$|\partial f/\partial y(t,y)| \leq \frac{2}{3(y_0/2)^{1/3}} \equiv K.$$

So by the Mean Value Theorem, given any $x, y \in J$ there is some z between x and y such that

$$\frac{|f(x) - f(y)|}{|x-y|} = |f_y(z)| \leq K$$

and therefore f is Lipschitz on J with constant K. Hence Theorem 1.1 implies the existence of a unique solution of $\frac{dy}{dt} = y^{2/3}$ $y(0) = y_0$ on some time interval.

◇

To prove the existence and uniqueness theorem, we need some machinery from real analysis.

2 Metric Spaces

A **metric space** is a set X, together with a distance function (or metric) $d : X \times X \to \mathbb{R}$ that satisfies the following conditions:

1. $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

Example 2.1. $X = \mathbb{R}^n$, together with the usual Euclidean distance $d(x,y) = |x-y|$ is a metric space.

◇

We say a sequence $\{x_k\}$ in a metric space X **converges** to x in X if $\lim_{k\to\infty} d(x_k, x) = 0$. That is, x_k converges to x if for every $\epsilon > 0$ there is some N such that $k > N$ implies $d(x_k, x) < \epsilon$. We then write $\lim_{k\to\infty} x_k = x$, or simply $x_k \to x$.

A sequence $\{x_k\}$ is called a **Cauchy sequence** if for every $\epsilon > 0$ there is some N such that $m, n > N$ implies $d(x_m, x_n) < \epsilon$. It is easy to see that convergent sequences are Cauchy sequences.

A metric space X is called **complete** if every Cauchy sequence converges to an element of X.

Example 2.2. $\mathbb{R}^n$ is complete.

$\diamond$

Example 2.3. $\mathbb{Q}$ is not complete. Let $\{x_n\}$ be any sequence in $\mathbb{Q}$ that converges to $\sqrt{2}$. We know such a sequence exists by the density of the rationals in the reals. Then $\{x_n\}$ is a Cauchy sequence, but does not converge to an element of $\mathbb{Q}$.

$\diamond$

3 Uniform Convergence and Spaces of Continuous Functions

The **uniform norm** (or **sup norm**) of a function f on an interval I is defined by

$$\|f\| = \sup_{x\in I} |f(x)|$$

Example 3.1. Let f be defined on $\mathbb{R}$ by $f(x) = \arctan(x)$. Since $|f(x)| < \pi/2$ for all $x \in \mathbb{R}$ and $\lim_{x\to\infty} f(x) = \pi/2$, it follows that $\|f\| = \pi/2$.

$\diamond$

Example 3.2. Let $f(x) = x^2$. On the interval $I = [-3, 3]$ the sup norm of f is $\|f\| = 9$. On $\mathbb{R}$, the sup norm of f is $\|f\| = \infty$ since f is unbounded on $\mathbb{R}$.

$\diamond$

Convergence with respect to the uniform norm is known as **uniform convergence**. We say a sequence of functions f_n **converges uniformly** to a function f on the interval I if

$$\lim_{n\to\infty} \|f_n - f\| = \lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

Example 3.3. Let $f_n(x) = x^n$ and let $f(x) = 0$. Then on the domain $[0, 1/2]$ we have

$$\|f_n - f\| = \sup_{x\in[0,1/2]} |x^n| = \left(\frac{1}{2}\right)^n \to 0$$

so f_n converges uniformly to f on the domain $[0, 1/2]$. However, on the domain $[0, 1]$ we have

$$\|f_n - f\| = \sup_{x\in[0,1]} |x^n| = 1 \not\to 0,$$

so f_n does not converge uniformly to f on $[0, 1]$.

$\diamond$

One important feature of uniform convergence is that it preserves continuity. That is, the uniform limit of a sequence of continuous functions is continuous.

Theorem 3.1. *Let $\{f_n\}$ be a sequence of continuous functions on some interval I in $\mathbb{R}$, and suppose that f_n converges uniformly on I to a function f. Then f is continuous on I.*

Proof. Fix $x \in I$, and let $\epsilon > 0$ be given. Then since f_n converges uniformly to f, we may choose n such that $\|f_n - f\| < \epsilon/3$. Since f_n is continuous at x, there exists some $\delta > 0$ such that for any y in I with $|y - x| < \delta$ we have $|f_n(y) - f_n(x)| < \epsilon/3$. Therefore, for any such y, we also have by the triangle inequality:

$$\begin{aligned}|f(y) - f(x)| &\leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)| \\ &\leq \|f - f_n\| + |f_n(y) - f_n(x)| + \|f_n - f\| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon\end{aligned}$$

Therefore f is continuous at x. Since this holds for every x in I, f is continuous on I. $\square$

Let $C([a, b], [c, d])$ denote the space of continuous functions $f : [a, b] \to [c, d]$. These are simply the continuous functions whose graph lies inside the rectangle $R = [a, b] \times [c, d]$. We can make $C([a, b], [c, d])$ into a metric space by setting

$$d(f, g) = \|f - g\|.$$

Theorem 3.2. *The space $C([a, b], [c, d])$ is complete.*

Proof. Let f_n be a Cauchy sequence in $C([a, b], [c, d])$. Then given $\epsilon > 0$ there is some N such that $\|f_m - f_n\| < \epsilon$ whenever $m, n > N$. For each fixed x in $[a, b]$, we have

$$|f_m(x) - f_n(x)| \leq \|f_m - f_n\| < \epsilon,$$

so the sequence of real numbers $\{f_n(x)\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, this sequence converges to some real number, which we shall call $f(x)$. Doing this for each x in $[a, b]$ defines a function f defined on $[a, b]$. Since each sequence $\{f_n(x)\}$ lies in the closed interval $[c, d]$, its limit $f(x)$ is also in $[c, d]$, and therefore the function f maps $[a, b]$ into $[c, d]$.

Next we show that f_n converges uniformly to f. Given $\epsilon > 0$, choose N so that $\|f_m - f_n\| < \epsilon/2$ whenever $m, n > N$. Then for any x in $[a, b]$, we have

$$\begin{aligned}|f_m(x) - f(x)| &\leq |f_m(x) - f_n(x)| + |f(n(x) - f(x)| \\ &\leq \|f_m - f_n\| + |f(n(x) - f(x)| \\ &< \epsilon/2 + |f_n(x) - f(x)|.\end{aligned}$$

Now since $f_n(x)$ converges to $f(x)$, it follows that $|f_n(x) - f(x)| < \epsilon/2$ for large enough n, and therefore we have

$$|f_m(x) - f(x)| < \epsilon$$

for any $m > N$. Since the choice of m did not depend on x, this inequality holds for every x in $[a, b]$. Therefore

$$\|f_m - f\| = \sup_{x \in I} |f_m(x) - f(x)| \leq \epsilon,$$

so f_m converges uniformly to f. By Theorem 3.1, it follows that the limit function f is continuous on $[a, b]$. We have therefore shown that any Cauchy sequence in $C([a, b], [c, d])$ converges to an element of $C([a, b], [c, d])$. Hence $C([a, b], [c, d])$ is complete. □

4 Fixed Point Iteration and Contraction Mappings

Let X be a metric space. A **fixed point** of a function $T : X \to X$ is an element $x \in X$ such that $T(x) = x$.

Example 4.1. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x^2$. Then $T(0) = 0$ and $T(1) = 1$, so T has fixed points $x = 0$ and $x = 1$ are fixed points.

◇

Example 4.2. Let $X = C([0, 1], [2, 3])$. Given a function $y \in X$, let

$$T(y)(t) = 2 + \frac{1}{3} \int_0^t y(s)\, ds$$

Then $T(y)$ is a continuous function, and for any $t \in [0, 1]$ we have

$$0 \leq \int_0^t y(s) \leq 3$$

and thus $2 \leq T(y)(t) \leq 3$. Hence $T(y) \in X$, so T is a map from X to X. Now let $y(t) = 2e^{\frac{1}{3}t}$. Then $y \in X$ and

$$T(y)(t) = 2 + \frac{1}{3} \int_0^t 2e^{\frac{1}{3}s}\, ds = 2 + 2e^{\frac{1}{3}s}\Big|_0^t = 2 + 2e^{\frac{1}{3}t} - 2 = 2e^{\frac{1}{3}t}.$$

In other words, $T(y) = y$, so the function y is a fixed point of T.

◇

A **contraction mapping** on a metric space X is a function $T : X \to X$ such that for some $\alpha < 1$,

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$. Thus, a contraction mapping with constant α shrinks the distance between distinct points by at least a factor of α.

Example 4.3. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x^2$. Then $|T(y) - T(x)| = |y + x||y - x|$. In order for T to be a contraction mapping, we need to restrict T to a domain X where $|y + x| \leq \alpha < 1$ for all $x, y \in X$. One such domain is $X = [-\frac{1}{4}, \frac{1}{4}]$, on which $\alpha = \frac{1}{2}$.

◇

Theorem 4.1. *(Contraction Mapping Principle) Let $T : X \to X$ be a contraction mapping on a complete metric space X. Then T has a unique fixed point $x \in X$.*

Proof. Let $x_0 \in X$ and define the sequence $\{x_k\}$ by setting

$$x_{k+1} = T(x_k)$$

for each $k \geq 0$. Set $d_0 = d(x_0, x_1)$. Then since

$$d(x_k, x_{k+1}) = d(T(x_{k-1}), T(x_k)) \leq \alpha d(x_{k-1}, x_k)$$

for $k \geq 1$, it follows by induction that $d(x_k, x_{k+1}) \leq \alpha^k d_0$. Now given $\epsilon > 0$ choose N so that $\alpha^N d_0/(1-\alpha) < \epsilon$. This can be done since $\alpha < 1$. Then for $m, n > N$ suppose without loss of generality that $m \leq n$. Then by the triangle inequality,

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k=m}^{n-1} \alpha^k d_0 \leq \sum_{k=m}^{\infty} \alpha^k d_0 = \frac{\alpha^m d_0}{1-\alpha} \leq \frac{\alpha^N d_0}{1-\alpha} < \epsilon.$$

Thus x_k is a Cauchy sequence which by completeness of X converges to some $x \in X$. Now $d(T(x_k), T(x)) \leq \alpha d(x_k, x) \to 0$ so $T(x_k)$ converges to $T(x)$. But $T(x_k) = x_{k+1}$ converges to x, so $T(x) = x$. To prove uniqueness, suppose also that $T(y) = y$. Then

$$d(x, y) = d(T(x), T(y)) \leq \alpha d(x, y)$$

which is possible only if $d(x, y) = 0$, which implies $x = y$. □

The process illustrated in the previous proof is known as **fixed point iteration**. The contraction mapping principle essentially says that, from any starting point x_0, the sequence of iterates of x_0 under a contraction mapping T will always converge to the unique fixed point of T.

5 Proof of the Existence and Uniqueness Theorem

We now proceed with the proof of Theorem 1.1. The idea is to first characterize solutions of the initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0 \tag{2}$$

as fixed points of a map on a complete metric space, and then show that, if the time interval is sufficiently small, this map is a contraction mapping. We begin by defining the map. Suppose f is continuous and $y(t)$ is a solution of (2) on some interval $(t_0 - \epsilon, t_0 + \epsilon)$. Then integrating from t_0 to t gives

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds. \tag{3}$$

This equation is called an integral equation, since it relates the unknown function y to an integral involving y. Thus any solution of (2) is a solution of the integral equation

(3). Conversely, suppose y is a continuous function which satisfies (3). Then $f(s, x(s))$ is continuous, and the Fundamental Theorem of Calculus implies that y is differentiable and $\frac{dy}{dt} = f(t, y(t))$. Furthermore, at $t = t_0$ we have $y(t_0) = y_0$. Thus any continuous solution of (3) is a solution of (2). Thus we now focus our attention on solving (2).

Given a continuous function y, we define $T(y)$ to be the function given by

$$T(y)(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Then y is a solution of (2) if and only if $T(y) = y$, i.e. y is a fixed point of the map T.

Next we wish to define a complete metric space on which T is a contraction mapping. We begin by defining for $\epsilon > 0$ and $\eta > 0$ the space

$$X = C([t_0 - \epsilon, t_0 + \epsilon], [y_0 - \eta, y_0 + \eta]).$$

By Theorem 3.2, X is complete.

Theorem 5.1. *Suppose $f(t, y)$ is continuous and Lipschitz with respect to y on the domain $R = [a, b] \times [c, d]$. Then for any (t_0, y_0) in R, there exist $\epsilon > 0$ and $\eta > 0$ such that $T : X \to X$ is a contraction mapping.*

Proof. First choose $\eta > 0$ small enough that the interval $[y_0 - \eta, y_0 + \eta]$ is contained within the interval $[c, d]$. Next, since f is Lipschitz with respect to y on R, there is some constant K such that $|f(t, y_2) - f(t, y_1)| \leq K|y_2 - y_1|$ for all (t, y_1) and (t, y_2) in R. Let

$$M = \max_{(t,y)\in R} |f(t, y)|.$$

Choose $\epsilon > 0$ such that $\epsilon < \min\{\frac{1}{K+1}, \frac{\eta}{M+1}\}$. We first show that T maps X to itself. Let $x(t)$ be a function in the space X, and let $y = T(x)$. Then

$$y(t) = y_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Since x is continuous and f is continuous, the composition $f(s, x(s))$ is continuous, so by the Fundamental Theorem of Calculus y is differentiable, and thus continuous. To prove that the range of y is a subset of $[y_0 - \eta, y_0 + \eta]$, observe that for $t_0 \leq t \leq t_0 + \epsilon$,

$$|y(t) - y_0| = \left| \int_{t_0}^{t} f(s, x(s))\, ds \right| \leq \int_{t_0}^{t} |f(s, x(s))|\, ds \leq \epsilon M < \frac{\eta M}{M+1} < \eta,$$

and similarly $|y(t) - y_0| < \eta$ for $t_0 - \epsilon \leq t < t_0$. Hence $y \in X$, so $T : X \to X$. To show that T is a contraction mapping with respect to the uniform norm, let $x, y \in X$, and denote

$\alpha = \epsilon K < 1$. Then for $t_0 \leq t \leq t_0 + \epsilon$,

$$\begin{aligned}
|T(x)(t) - T(y)(t)| &= \left| \int_{t_0}^{t} f(s, x(s)) - f(s, y(s))\, ds \right| \\
&\leq \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))|\, ds \\
&\leq K \int_{t_0}^{t} |x(s) - y(s)|\, ds \\
&\leq K \int_{t_0}^{t} \|x - y\|\, ds \\
&\leq K\epsilon \|x - y\| = \alpha \|x - y\|
\end{aligned}$$

Likewise $|T(x)(t) - T(y)(t)| \leq \alpha \|x - y\|$ for $t_0 - \epsilon \leq t < t_0$, so $\|T(x) - T(y)\| \leq \alpha \|x - y\|$ and therefore T is a contraction mapping. □

We now give the proof of Theorem 1.1.

Proof. By Theorem 3.2, X is a complete metric space with respect to the uniform norm. By Theorem 5.1, $T : X \to X$ is a contraction mapping provided ϵ and η are chosen sufficiently small. Thus by the Contraction Mapping Principle, there exists a unique fixed point y of T in X. The function y is the unique solution of the initial value problem (2) on the interval $[t_0 - \epsilon, t_0 + \epsilon]$. □

6 Picard Iteration

Now let us look back to the proof of the contraction mapping principle. In it, we found that the fixed point of T is the limit of $T^k(x_0)$, where x_0 is any element of X. Hence the solution of the initial value problem (2) can be found by iterating the function

$$T : y(t) \mapsto y_0 + \int_{t_0}^{t} f(s, y(s))\, ds$$

on any arbitrarily chosen continuous function satisfying the initial data. One natural choice is the constant function $x_0(t) \equiv y_0$. Then for $k \geq 0$ we define $x_{k+1} = T(x_k)$. That is,

$$x_{k+1}(t) = y_0 + \int_{t_0}^{t} f(s, x_k(s))\, ds.$$

This process is called **Picard iteration**.

Example 6.1. Consider the linear equation $\frac{dy}{dt} = ky$, $y(0) = y_0$. Picard iteration, with

initial function $x_0(t) \equiv y_0$ gives

$$\begin{aligned}
x_1(t) &= y_0 + \int_0^t k y_0 \, ds = (1 + tk) y_0 \\
x_2(t) &= y_0 + \int_0^t (1 + tk) y_0 \, ds = \left(1 + tk + \frac{1}{2} t^2 k^2\right) y_0 \\
&\vdots \\
x_k(t) &= \left(\sum_{j=0}^{k} \frac{t^j k^j}{j!}\right) y_0.
\end{aligned}$$

These are precisely the partial sums of the Taylor series for $e^{tk} x_0$, the unique solution.

$\diamond$