Solutions of Exercises page 33

2. المجموعة الوحيدة التي ليست مفتوحة وليست مغلقة في التمرين رقم (1)
هي في الفرع (e) وذلك بسبب $z=0$ تعتبر نقطة حدودية لكنه ليس
ضمن المجموعة (فعند الكلام عن الإحداثيات القطبية Poler (θ, r) فنستثني
$z=0$ لأن θ غير معرّفة عنده) فالمجموعة ليست مغلقة. والمجموعة ليست
مفتوحة لأنها تحوي على بعض نقاطها الحدودية مثل $z=2e^{i\frac{\pi}{4}}$.

3. المجموعة في الفرع (a) فقط هي bounded لإمكانية احتواء دائرة
عليها مثلاً $|z-2|=3$.

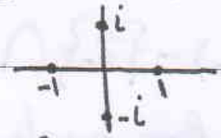
6. $(\Longrightarrow)$ S is open set, let $z \in S$

$\therefore$ z is not boundary point of S [open mean $\text{Boun}(S) \cap S = \phi$ def. of open set]

Now since $z \in S$, then for any nbd of z
$N(z)$, $N(z) \cap S \neq \phi$, then z not exterior point of S [def of exterior]
So z must be interior point of S.

$(\Longleftarrow)$ let $z \in S$, so z is interior point of S, then
$\therefore$ z not boundary point of S, [definition of boundary pt.]
that is, S contains none of its boundary points, so S is open set [by def.]

7. (a) $S = \{z_n \mid z_n = i^n, n = 1, 2, \dots\}$



$1 = i^4 = i^8 = \dots$
$-1 = i^2 = i^6 = \dots$

$= \{i, -1, -i, 1\}$ finite has no accumulation pt.

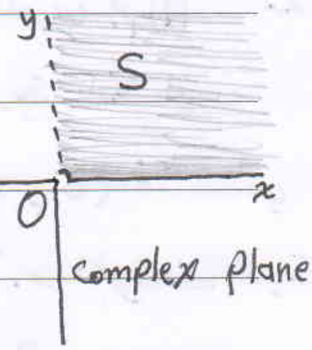
(b) $S = \{z_n \mid z_n = \frac{i^n}{n}, n = 1, 2, \dots\}$

$= \{i, \frac{-1}{2}, \frac{-i}{3}, \frac{1}{4}, \frac{i}{5}, \frac{-1}{6}, \frac{-i}{7}, \frac{1}{8}, \dots\} \to 0$

$z_0 = 0$ is an accumulation point of S because for each deleted nbd of 0 $N(0) - \{0\}$ $(0 < |z| < \varepsilon)$ contains at least one point in S.

(c) $0 \leq \arg z < \frac{\pi}{2}$ $(z \neq 0)$

$S = \{ re^{i\theta} \mid 0 \leq \theta < \frac{\pi}{2}, r \neq 0 \}$



the set of all accumulation points of S

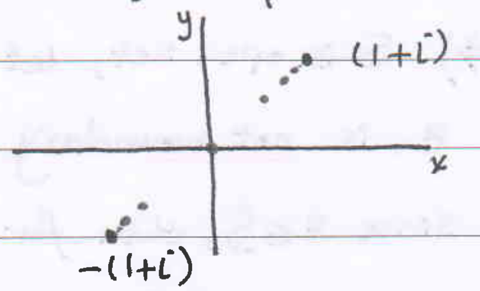
$= \{ z = x + iy \mid x \geq 0 \text{ and } y \geq 0 \}$

$= S \cup \{ z \mid z = x + iy ;\ x = 0 \text{ and } y > 0 \}$

(d) $z_n = (-1)^n (1+i) \frac{n-1}{n}, (n = 1, 2, \ldots)$

$S = \{ 0, \frac{1}{2}(1+i), \frac{-2}{3}(1+i), \frac{3}{4}(1+i), \frac{-4}{5}(1+i), \frac{5}{6}(1+i), \ldots \}$

There are two accumulation points of S $\{(1+i), -(1+i)\}$.



8. S contains each accumulation point of it.

Let z_0 be an arbitrary boundary point of S,

So for any nbd of z_0 $N(z_0) \cap S \neq \phi$ and $N(z_0) \cap S^c \neq \phi$.

Assume that $z_0 \notin S$, then

$N(z_0) - \{z_0\} \cap S \neq \phi$,

that is, each deleted neighborhood of z_0 contains at least a point in S,

i.e. z_0 is an accumulation point of S, then $z_0 \in S$ [by hypothesis], which is contradiction to our assumption ($z_0 \notin S$), then $z_0 \in S$ (i.e. S contains all its boundary points). Thus S is closed [by def.].

8. تنبيه: Infact the converse of exercise 8 is true (i.e. a closed set contains all its accumulation points), to see that, let z_0 be an accumulation point of S (closed set)

So each deleted neighborhood of z_0 $N(z_0)-\{z_0\}\cap S\neq\phi$,

assume $z_0\notin S$, (i.e $z_0\in S^c$)

$$N(z_0)\cap S\neq\phi \text{ and } N(z_0)\cap S^c\neq\phi$$

$\therefore$ z_0 is a boundary point of S [by def. of boundary]

so $z_0\in S$ [by hypothesis, S is closed set]

but this is contradiction to our assumption ($z_0\notin S$),

then must $z_0\in S$, and so S contains each accumulation points of it.

9. Let D be a domain, and

$z_0\in D$

Since D is open, then z_0 is an interior point of D

[by exercise 6]

So $\exists$ nbd of z_0, $N(z_0)\subseteq D$,

for any deleted neighborhood of z_0, $dnbd(z_0)$,

$dnbd(z_0)\cap N(z_0)\neq\phi$,

So $dnbd(z_0)\cap D\neq\phi$

Thus z_0 is an accumulation point of D.

Remark. $dnbd(z_0)=\{z \mid 0<|z-z_0|<\varepsilon, \text{for some } \varepsilon>0\}$

10. Let $S = \{z_1, z_2, \ldots, z_n\}$,

assume that z_0 is an accumulation point of S,
take $\delta = \min_i |z_0 - z_i|$ where $i = 1, 2, \ldots, n$ if $z_0 \notin S$,
and $i = 1, 2, \ldots, k-1, k+1, \ldots, n$ if $z_0 \in S$
(for some k s.t. $z_0 = z_k$)

Now the deleted neighborhood

$$0 < |z - z_0| < \varepsilon = \frac{\delta}{2} \quad (\text{dnbd}(z_0))$$

has no point in S (i.e. $\exists$ $\text{dnbd}(z_0) \cap S = \phi$),
this is controduction to the definition of accumulation point. Thus S has no accumulation point.