MATH645: Complex Analysis
Lecture Notes

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Contents

LECTURE 1

The Complex Numbers: Basic Concepts

The real numbers, while we can easily imagine them, do not fully serve ever increasing demand of the modern science. The main deficiency is that not every polynomial has a real root. For example, the simple quadratic binomial $x^2 + 1$ has no root in the set of real numbers $\mathbb{R}$. However, one notices that $x = \pm\sqrt{-1}$ formally solve the equation $x^2 + 1 = 0$. While similar formal computations were done centuries ago everyone would refuse this by a very simple reason: we cannot imagine the number $\sqrt{-1}$ hence it does not exist. It will be rigorously proven in this lesson that $\sqrt{-1}$ can be properly introduced by expanding to a larger number system, the set of the so-called complex numbers $\mathbb{C}$. We will also show in our course that every polynomial then has a solution in $\mathbb{C}$. The latter has a tremendous impact on math and its applications. In this lesson we will formally define the complex numbers. Although complex numbers were likely introduced to you before differently the content of this lesson should be very easy.

Definition 1.1. *The set of complex numbers $\mathbb{C}$ is the set of all ordered pairs (a, b), where $a, b \in \mathbb{R}$ equipped with the following two operations: addition $(+)$ and multiplication $(\cdot)$, where for every $a, b, c, d \in \mathbb{R}$:*

$$\begin{aligned} (a, b) + (c, d) &= (a + c, b + d) \\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc). \end{aligned} \tag{1.1}$$

Proposition 1.2. *The set of complex numbers $\mathbb{C}$ is a commutative field.*

The proof is left as an exercise.

Note that the subset $\{(a, 0)|\ a \in \mathbb{R}\}$ of $\mathbb{C}$ can be identified with the set of real numbers $\mathbb{R}$. By convention we agree to write $(1, 0) = 1$ and hence $(a, 0) = a$. Similarly, the complex number $(0, 1)$, usually denoted i (engineers and physicists like j instead as they use i for the electric current), is called the imaginary unit.

One has

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1$$

which you have likely seen as a definition of the imaginary unit.

For any complex number $z = (a, b)$ we can write

$$z = (a, b) = (a, 0) + (0, b) = a\,(1, 0) + (b, 0)(0, 1) = a + bi. \tag{1.2}$$

Definition 1.3. *Given a complex number $z = (a, b)$, a is called the real part of z, and b is called the imaginary part of z. We write*

$$a = \operatorname{Re} z\ , \quad b = \operatorname{Im} z.$$

Using Definition 1.3 equation (1.2) reads

$$z = \operatorname{Re} z + i \operatorname{Im} z \tag{1.3}$$

which is the so-called standard form of a complex number.

Remark 1.4. *The reader may observe that in* (1.2) *and* (1.3) *we put i in different positions. Either one is acceptable and it depends on the particular situation where we want to put our i. A general rule is if we have an imaginary part that is a variable we put i before the variable and if we have an imaginary part that is a numerical coefficient we put i after it. Hopefully, you will soon get used to this convention.*

Definition 1.5. *Given $z = a + ib$, we define its (complex) conjugate $\overline{z}$ by*

$$\overline{z} := a - ib$$

and its modulus (absolute value) $|z|$ by

$$|z| := \sqrt{z\overline{z}}. \tag{1.4}$$

Since $\mathbb{C}$ is a field, every non-zero z has a multiplicative inverse z^{-1}.

Proposition 1.6. *Let $z \in \mathbb{C}$ and $z \neq 0$. Then*

$$z^{-1} = \left(\frac{\operatorname{Re} z}{|z|^2}, \frac{-\operatorname{Im} z}{|z|^2} \right) = \frac{\overline{z}}{|z|^2}. \tag{1.5}$$

Note that with (1.5) in hand, we can define division

$$\frac{z_1}{z_2} := z_1 . z_2^{-1} \qquad \forall z_1, z_2 \in \mathbb{C} \ , \ z_2 \neq 0.$$

Proposition 1.7. *Conjugates and moduli have the following properties:*

(1) $\operatorname{Re} z = \dfrac{z + \overline{z}}{2}$, $\operatorname{Im} z = \dfrac{z - \overline{z}}{2i}$

(2) $|z| = \sqrt{\operatorname{Re}^2 z + \operatorname{Im}^2 z} \geq 0$ and $|z| = 0 \iff z = 0$

(3) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(4) $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$

(5) $\overline{\left(\dfrac{z_1}{z_2} \right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$

(6) $|z_1 z_2| = |z_1| \, |z_2|$

(7) $\left| \dfrac{z_1}{z_2} \right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \neq 0$

(8) $z = \overline{z} \iff z \in \mathbb{R}$

(9) $|\overline{z}| = |z|$.

Note that $\mathbb{C}$ can be identified with $\mathbb{R}^2$ and hence we can call $\mathbb{C}$ the complex plane. Then we can graphically associate to any complex number z a point in the complex plane. Observe that the distance $d(z_1, z_2)$ between two points $z_1, z_2 \in \mathbb{C}$ can be computed by the formula:

$$d(z_1, z_2) = |z_1 - z_2| .$$

The following statement will play a fundamental role in our course.

THEOREM 1.8 (Triangle Inequality). *For all* $z_1, z_2 \in \mathbb{C}$

$$|z_1 + z_2| \leq |z_1| + |z_2| . \tag{1.6}$$

COROLLARY 1.9. *For all* $z_1, z_2 \in \mathbb{C}$

$$|z_1 - z_2| \geq ||z_1| - |z_2|| \tag{1.7}$$

and hence combining (1.6) *and* (1.7) *one has the two sided triangle inequality*

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2| .$$

Before we conclude this section we should take a look at the graphical representation of some of the topics we have gone over.

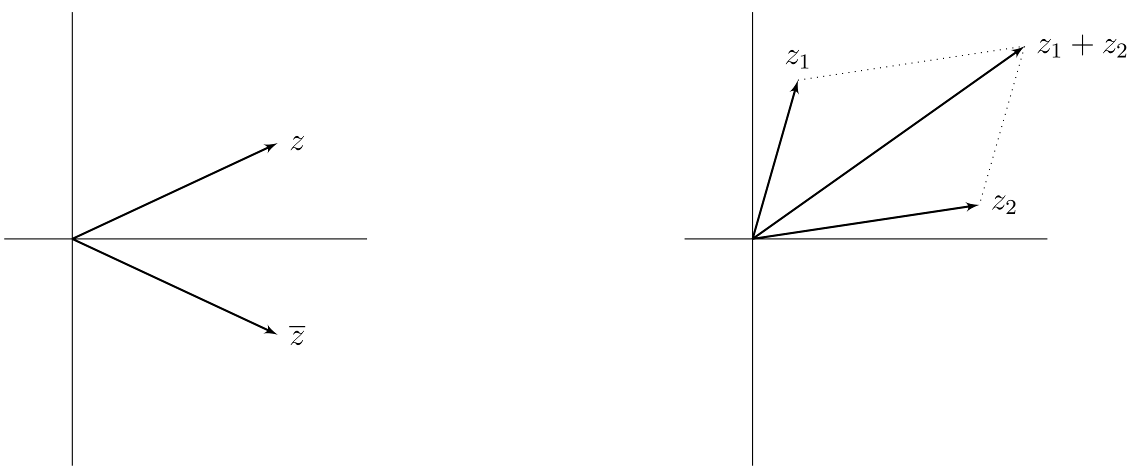


FIGURE 1. Graphical interpretation of conjugates and sums

Exercises

Exercise 1.1 Prove Proposition 1.2.

Exercise 1.2 Prove Proposition 1.6.

Exercise 1.3 Prove Proposition 1.7.

Exercise 1.4 Prove Theorem 1.8 and Corollary 1.9.

LECTURE 2

Limits, Series, Sets, etc.: Basic Concepts

Basic concepts of Real Analysis are easily adaptable to help us construct our theory of Complex Analysis. The main concept we will use from our study of Real Analysis is the concept of the limit of a sequence. We need to define a few other concepts before we can define the limit.

1. Definitions for Limit and Convergence

Recall in Real Analysis the concept of the ε-neighborhood of a point x_0. For some $\varepsilon > 0$, it is the set $U_\varepsilon := \{x : \ |x - x_0| < \varepsilon\}$.

In Complex Analysis such a concept is represented in the complex plane by a disk, hence the following notation and definition.

NOTATION. *We denote*

$$\mathbb{D}_r(z_0) = \{z \mid |z - z_0| < r\}$$

the disk of radius r *centered at* z_0*, and* $\mathbb{D} := \mathbb{D}_1(0)$*, the unit disk.*

DEFINITION 2.1. *Let* $\varepsilon > 0$ *and* $z_0 \in \mathbb{C}$*. The set* $\mathbb{D}_\varepsilon(z_0) = \{z \mid |z - z_0| < \varepsilon\}$ *is called the* <u>*ε-neighborhood*</u> *of a point* z_0*. We also define the* <u>*deleted neighborhood*</u> *of a point* z_0 *or* <u>*punctured disk*</u> *of center* z_0*:*

$$\mathring{\mathbb{D}}_\varepsilon(z_0) = \mathbb{D}_\varepsilon(z_0) \setminus \{z_0\}.$$

DEFINITION 2.2. *A* <u>*sequence*</u> *is defined to be an ordered, countably infinite collection of elements from* $\mathbb{C}$*. For a sequence in the complex plane we write* $\{z_n\}_{n\geq 1}$ *where* $\{z_n\}_{n\geq 1} = \{z_1, z_2, ..., z_n, ...\}$.

DEFINITION 2.3. *A sequence* $\{z_n\}_{n\geq 1}$ *is said to* <u>*converge*</u> *to a number* $z \in \mathbb{C}$ *if for all* $\varepsilon > 0$ *there exists some* $N_\varepsilon \in \mathbb{N}$ *such that* $n > N_\varepsilon \Rightarrow z_n \in \mathbb{D}_\varepsilon\,(z)$.

The complex number z is called the <u>limit</u> of the sequence $\{z_n\}_{n\geq 1}$ and is usually written

$$\lim_{n\to\infty} z_n = z. \tag{2.1}$$

REMARK 2.4. *Note that the limit can also be expressed as follows:*

$$z = \lim z_n, \ n \to \infty$$

$$z_n \to z, \ n \to \infty$$

$$z_n \underset{n\to\infty}{\longrightarrow} z.$$

Definition 2.3 can be stated in several different ways utilizing the different methods to express the idea of convergence. For practice you should restate the definition using each method for expressing the idea of convergence.

The geometrical meaning of Definition 2.3 can easily be seen in Figure 1.

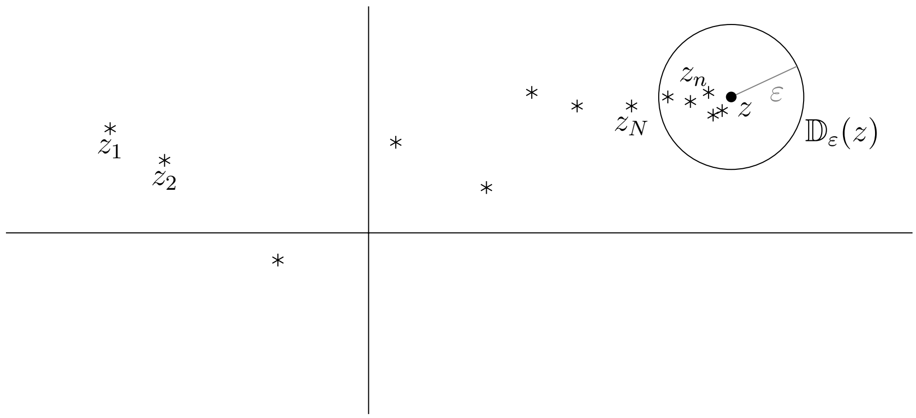


FIGURE 1. A sequence $\{z_n\}$ converging to z

DEFINITION 2.5. *We say that* $\{z_n\}_{n\geq 1}$ *<u>diverges</u> (or is divergent) if* $\{z_n\}_{n\geq 1}$ *does not converge for any* $z \in \mathbb{C}$.

PROPOSITION 2.6. *Given* $\{z_n\}_{n\geq 1}$,

$$z_n \to z \quad \Longleftrightarrow \quad \operatorname{Re} z_n \to \operatorname{Re} z \text{ and } \operatorname{Im} z_n \to \operatorname{Im} z.$$

PROOF. Assume that $z_n \to z$. Then

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \ : \ |z_n - z| < \varepsilon \quad \forall n > N_\varepsilon.$$

But by the triangle inequality we have that

$$|\operatorname{Re}(z_n - z)| \leq |z_n - z| < \varepsilon.$$

That is Re $z_n \to \operatorname{Re} z$ as $n \to \infty$. Similarly one shows that $\operatorname{Im} z_n \to \operatorname{Im} z$. Conversely, assume that (abbreviating $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$)

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \ : \ |x_n - x|, |y_n - y| < \frac{\varepsilon}{2} \quad \forall n > N_\varepsilon.$$

Then by the triangle inequality

$$|z_n - z| \leq |x_n - x| + |y_n - y| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

DEFINITION 2.7. *A given sequence* $\{z_n\}_{n\geq 1}$ *is said to be <u>Cauchy</u> (a Cauchy sequence) if*

$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{N} \ : \quad \forall m, n \geq N_\varepsilon \quad |z_m - z_n| < \varepsilon.$$

THEOREM 2.8 (Cauchy's Criterion). *A given sequence* $\{z_n\}_{n\geq 1}$ *is convergent if and only if* $\{z_n\}_{n\geq 1}$ *is a Cauchy sequence.*

The proof is left as an exercise.

DEFINITION 2.9. *The formal sum* $\sum_{n\geq 1} z_n$ *of the elements of the sequence* $\{z_n\}_{n\geq 1}$ *is called a* *series*.

DEFINITION 2.10. *Let* $\sum_{n\geq 1} z_n$ *be a series. Then* $S_n := \sum_{k=1}^{n} z_k$ *is called a partial sum. A series is called convergent if its sequence of partial sums is convergent. In this case we write*

$$\sum_{k\geq 1} z_k = \lim_{n\to\infty} S_n = S.$$

DEFINITION 2.11. *A series which does not converge is called divergent.*

DEFINITION 2.12. *A given series* $\sum_{n\geq 1} z_n$ *is said to be absolutely convergent if* $\sum_{n\geq 1} |z_n|$ *converges.*

2. Complex Sets

We will now turn our attention to complex sets.

DEFINITION 2.13. *A set* $E \subseteq \mathbb{C}$ *is called open if for all* $z \in E$, $\exists\, \mathbb{D}_\varepsilon(z)$ *with some* $\varepsilon > 0$ *such that* $\mathbb{D}_\varepsilon(z) \subset E$.

In other words, every point of an open set $E \subset \mathbb{C}$ comes with a neighborhood contained in E.

Note that by Definition 2.1 an ε-neighborhood is an open set, and hence we refer to $\mathbb{D}_r(z_0)$ as the open disk of radius r and of center z_0.

DEFINITION 2.14. *A set* $E \subseteq \mathbb{C}$ *is called connected if for all* $z_0, z_1 \in E$, *there is a continuous path* Γ *such that* $z_0, z_1 \in \Gamma$ *and* $\Gamma \subset E$.

In other words, any pair of points from a connected set can be connected with a continuous curve lying entirely in the set.

DEFINITION 2.15. *A point* $z \in \mathbb{C}$ *is called a boundary point of a set* E *if for all* $\varepsilon > 0$, $\mathbb{D}_\varepsilon(z) \cap E \neq \emptyset$ *and* $\mathbb{D}_\varepsilon(z) \cap (\mathbb{C}\backslash E) \neq \emptyset$. *We use* ∂E *to denote the set of all boundary points of a set* $E \subseteq \mathbb{C}$.

DEFINITION 2.16. *A set* E *is called closed if* $\partial E \subseteq E$.

Note that by definition $E \cup \partial E$ is always closed. Given a set E, the procedure of adding the boundary to the set is called the closure of E and denoted by either $\overline{E}$ or $\operatorname{clos} E$. That is

$$\overline{E} = \operatorname{clos} E = E \cup \partial E.$$

EXAMPLE 2.17.

(1) *Let* $E = \{z \in \mathbb{C} : \ |z| \neq 0\}$. *Then the set* E *is open and the boundary of* E *is the set* $\{0\}$.

(2)

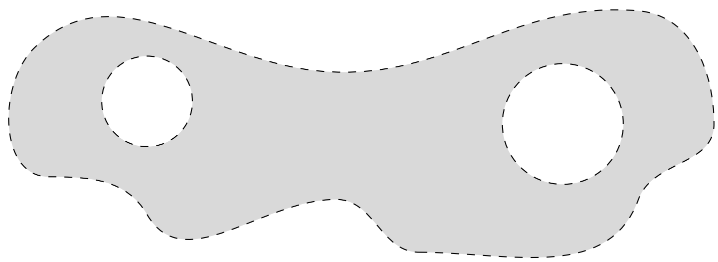


FIGURE 2. An open, connected set

(3) $\operatorname{clos}\{z : \ |z| \neq 0\} = \mathbb{C}$.

(4) $\overline{\mathbb{D}}_\varepsilon(z_0) = \{z : \ |z - z_0| \leq \varepsilon\}$

DEFINITION 2.18. *A point* z *of a set* E *is called an* *interior* *point of* E *if* z *is not contained in the boundary of* E.

3. Complex Functions

We conclude this lecture with a discussion of complex functions.

DEFINITION 2.19. *Given a set* $E \subseteq \mathbb{C}$, *a* *complex valued function* $f(z)$ *is any function of a complex variable* z *that maps* E *to* $\mathbb{C}$.

Given $z \in \mathbb{C}$ where $z = x + iy$, we can always write

$$f(z) = u(x, y) + iv(x, y)$$

where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

DEFINITION 2.20. *Let* E *be an open set and let* $f : E \to \mathbb{C}$ *be a complex valued function. We write*

$$\lim_{z \to z_0} f(z) = \ell$$

if $$\forall\ \varepsilon > 0 \quad \exists\ \delta_\varepsilon > 0\ :\ z \in \mathring{\mathbb{D}}_{\delta_\varepsilon}(z_0) \cap E \quad \Rightarrow \quad f(z) \in \mathbb{D}_\varepsilon(\ell).$$

DEFINITION 2.21. *Given an open set* E, *a function* $f : E \to \mathbb{C}$ *is said to be continuous at* $z_0 \in E$ *if*

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Assuming that the limit exists and $f(z_0)$ is defined.

DEFINITION 2.22. *If* $f : E \to \mathbb{C}$ *is continuous at each* $z \in E$ *then we say that* f *is a* *continuous function* *on* E.

Exercises

Exercise 2.1 Assuming Cauchy's Criterion for real sequences, prove Theorem 2.8.

Exercise 2.2 Prove that $\sum_{k\geq 1} z_k$ converges if and only if $\sum_{k\geq n} z_k \to 0, n \to \infty$.

Exercise 2.3 Is $\left\{\frac{i}{n}\right\}_{n\geq 1}$ open or closed? What is $\operatorname{clos}\left(\left\{\frac{i}{n}\right\}_{n\geq 1}\right)$?

Exercise 2.4 Let $f, g : E \to \mathbb{C}$ such that f and g are continuous at z_0. Prove the following facts:

(1) $f + g$ is continuous at z_0
(2) fg is continuous at z_0
(3) $\dfrac{f}{g}$ is continuous at z_0 if $g(z_0) \neq 0$.

LECTURE 3

Uniform Convergence of Sequences and Series

Consider a sequence $\{f_n(z)\}$ in which the elements are functions $f_n : E \to \mathbb{C}$. Such sequences are called functional sequences and play a crucial role in analysis. Let $C(E)$ denote the set of all continuous complex-valued functions on a set E. We say that a functional sequence $\{f_n(z)\}$, $f_n : E \to \mathbb{C}$ converges pointwise if for every $z_0 \in E$, the numerical sequence $\{f_n(z_0)\}$ converges. You may recall from Real Analysis a notion of almost everywhere convergence that is weaker than pointwise convergence. In our course, we will not be needing this idea.

EXAMPLE 3.1. *Some examples from Real Analysis:*

(1) $\{f_n(x)\}$ *on* $(0,1)$ *as defined in Figure 1:*

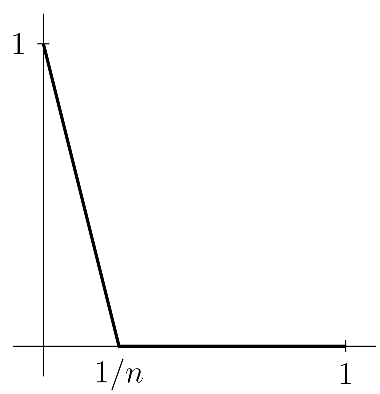


FIGURE 1. A real-valued sequence of functions converging pointwise to 0

We can see that $f_n \in C(0,1)$ *for all* n *and that* f_n *converges pointwise to zero as* n *goes to infinity. (On* $[0,1]$*, however,* $f_n \to 0$ *almost everywhere.)*

(2) $\left\{\dfrac{n}{n+1}x\right\}$ *on* $(0,1)$ *converges to* x *pointwise.*

(3) $\{\sin(nx)\}$ *on* $(0,\pi)$ *diverges pointwise.*

The following definition is very important to our course.

DEFINITION 3.2. *A sequence* $\{f_n(z)\}$, $f_n : E \to \mathbb{C}$ *is said to converge uniformly to a function* $f : E \to \mathbb{C}$ *if*

$$\sup_{z\in E} |f(z) - f_n(z)| \to 0 \text{ as } n \to \infty,$$

and we write $f(z) = \text{u-lim}\, f_n(z)$ *or* $f_n \rightrightarrows f$ *on* E.

Recall that "sup" stands for supremum. In our course, supremum will often be the absolute maximum.

For convenience, we will use the following shorthand notation:

$$\|f\|_\infty := \sup_{z\in E} |f(z)|$$

when it is clear from the context what E is.

REMARK 3.3.

(1) *In Definition 3.2 the set E need not be open or connected.*
(2) *Uniform convergence implies pointwise convergence but pointwise convergence does not imply uniform convergence.*

The following important theorem justifies the concept of uniform convergence:

THEOREM 3.4. *Let E be an open set in $\mathbb{C}$ and $\{f_n(z)\}$ be a sequence of continuous functions on E. Then $f_n \rightrightarrows f$ on E implies $f \in C(E)$.*

PROOF. Let $\varepsilon > 0$ be given. Thus, $f_n \rightrightarrows f$ on E implies that there exists an $N > 0$ such that

$$\|f - f_N\|_\infty < \varepsilon/3. \tag{3.1}$$

Next, $f_N \in C(E)$, so f_N is continuous at every $z_0 \in E$. Therefore, there exists some $\delta > 0$ such that for every $z \in \mathbb{D}_\delta(z_0)$

$$|f_N(z) - f_N(z_0)| < \varepsilon/3. \tag{3.2}$$

Now, take N as in (3.1) and z as in (3.2) and consider:

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_N(z) + f_N(z) - f_N(z_0) + f_N(z_0) - f(z_0)| \\ &\le \underbrace{|f(z) - f_N(z)|}_{\substack{\le \|f-f_N\|_\infty < \varepsilon/3 \\ \text{by (3.1)}}} + \underbrace{|f_N(z) - f_N(z_0)|}_{\substack{< \varepsilon/3 \\ \text{by (3.2)}}} + \underbrace{|f_N(z_0) - f(z_0)|}_{\substack{\le \|f-f_N\|_\infty < \varepsilon/3 \\ \text{by (3.1)}}} \end{aligned}$$

Therefore,

$$|f(z) - f(z_0)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, for all $\varepsilon > 0$ we have found $\delta > 0$ such that for all $z \in \mathbb{D}_\delta(z_0)$, $|f(z)-f(z_0)| < \varepsilon$. Therefore, f is continuous at z_0. But since z_0 was arbitrarily chosen, we know then that f is continuous at every $z \in E$. Therefore, f is a continuous function on E. $\square$

We turn now to functional series. We say that a series $\displaystyle\sum_{n\ge1} f_n(z)$ converges pointwise on a set E if the sequence $\left\{\displaystyle\sum_{k\le n} f_k(z)\right\}$ converges pointwise on E.

DEFINITION 3.5. $\displaystyle\sum_{n\ge1} f_n(z)$ *converges uniformly on E if the sequence* $\left\{\displaystyle\sum_{k\le n} f_k(z)\right\}$ *converges uniformly on E.*

By Theorem 3.4, if $f_n(z) \in C(E)$ for every n, and $\sum_{n\geq 1} f_n(z)$ converges uniformly, then

$$f(z) := \sum_{n\geq 1} f_n(z) \in C(E).$$

The problem with this statement is that verifying convergence is usually very difficult. The following theorem helps a lot.

THEOREM 3.6 (Weierstrass M-Test). *If* $\sup_{z\in E} |f_n| \leq M_n < \infty$ *and* $\sum_{n\geq 1} M_n < \infty$, *then* $\sum_{n\geq 1} f_n(z)$ *is uniformly convergent on* E.

PROOF. Consider the sequence $\{S_n(z)\}$ of partial sums, with $S_n(z) = \sum_{k=1}^{n} f_k(z)$. Let $n > m$, and notice

$$\begin{aligned} |S_n(z) - S_m(z)| &= \left| \sum_{k=m+1}^{n} f_k(z) \right| \\ &\leq \sum_{k=m+1}^{n} |f_k(z)| \\ &\leq \sum_{k=m+1}^{n} M_k \\ &= \left| \sum_{k=1}^{n} M_k - \sum_{k=1}^{m} M_k \right| \end{aligned} \tag{3.3}$$

Since $\sum_{k\geq 1} M_k < \infty$, then $\left\{ \sum_{k=1}^{n} M_k \right\}$ is a Cauchy sequence. It follows from (3.3) that $\{S_n(z)\}$ is Cauchy for any $z \in E$. By the Cauchy Criterion (Theorem 2.8) $\{S_n(z)\}$ converges to some $S(z)$.

It remains to show that $S_n \rightrightarrows S$ on E. Notice

$$\begin{aligned} \|S - S_n\|_\infty &= \left\| \sum_{k\geq n+1} f_k(z) \right\|_\infty \\ &\leq \sum_{k\geq n+1} M_k \end{aligned} \tag{3.4}$$

Since $\sum_{k\geq 1} M_k < \infty$, then the right hand side of (3.4) converges to 0 as $n \to \infty$. Therefore, as $n \to \infty$, $\|S - S_n\|_\infty \to 0$. □

Exercises

Exercise 3.1 Show that

(1) The sequence $\{f_n\}$ in Example 3.1(1) converges pointwise to 0, on $(0,1)$, but doesn't converge uniformly. Does $\{f_n\}$ converge on a subset of $(0,1)$?
(2) The sequence in Example 3.1(2) converges uniformly to x.
(3) The sequence $\{\sin(nx)\}$ from Example 3.1(3) diverges pointwise on $(0,\pi)$, but $\{\|\sin(nx)\|_\infty\}_{n\geq 1}$ converges.
(4) Give an example of a sequence of continuous functions which converges pointwise, but the limiting function is discontinuous.

Exercise 3.2 Show that if $\sum f_k(z)$ converges absolutely, then $\sum f_k(z)$ converges. Also show that the converse is not true.

LECTURE 4

Power Series: Domain of Convergence

A particularly important type of functional series is a power series

$$\sum_{n\geq 0} a_n(z-z_0)^n \tag{4.1}$$

where $a_n, z_0 \in \mathbb{C}$.

We call the set of all $z \in \mathbb{C}$ for which (4.1) converges the domain of convergence.

You probably remember from calculus that for a real power series (4.1) the domain of convergence is an interval centered at z_0. The interval could be just one point (e.g. for the series $\sum_{n\geq 0} n!x^n$) or be the whole real line (as e.g. with $\sum_{n\geq 0} \frac{x^n}{n!}$).

The half-length of the interval of convergence is called the radius of convergence.

You should also remember some criteria of convergence.

Lemma 4.1. *Let $q \in \mathbb{R}$ and consider the series $\sum_{n\geq 0} q^n$. If $|q| < 1$ then $\sum_{n\geq 0} q^n$ converges to $\frac{1}{1-q}$. If $|q| \geq 1$ then $\sum_{n\geq 0} q^n$ diverges.*

The proof is left as an exercise.

We are now going to extend all of these ideas from calculus to complex power series.

Theorem 4.2 (Abel's Theorem). *If a power series $\sum_{n\geq 0} a_n(z-z_0)^n$ converges at a point $z_1 \neq z_0$ then it absolutely converges on the open disk $\mathbb{D}_{|z_1-z_0|}(z_0)$. Moreover, it uniformly converges on any closed disk $\overline{\mathbb{D}}_\rho(z_0)$ where $\rho < |z_1 - z_0|$.*

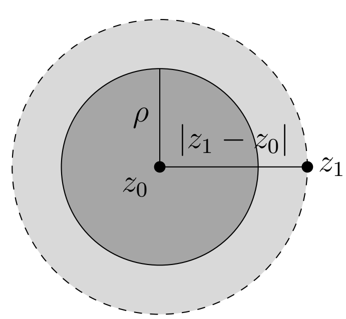


Figure 1. The open disk $\mathbb{D}_{|z_1-z_0|}(z_0)$ and the closed disk $\overline{\mathbb{D}}_\rho(z_0)$

Proof. Without loss of generality, set $z_0 = 0$. Take $z \in \mathbb{D}_{|z_1|}(0)$ and consider the series $\sum_{n\geq 0} a_n z^n$. Since z is an interior point of $\mathbb{D}_{|z_1|}(0)$, there is a q such that $0 < q < 1$ where

$$|z| = q|z_1|. \tag{4.2}$$

By our condition, $\sum_{n\geq 0} a_n z_1^n$ converges. This implies $a_n z_1^n \to 0$ as $n \to \infty$. Hence there exists a constant $M > 0$ such that $|a_n||z_1|^n \leq M$ and

$$|a_n z^n| = \underbrace{|a_n||z_1|^n}_{\leq M} \cdot \left|\frac{z}{z_1}\right|^n \leq M \left|\frac{z}{z_1}\right|^n. \tag{4.3}$$

Consequently it follows from (4.3) that

$$\begin{aligned}\left|\sum_{n\geq 0} a_n z^n\right| \leq \sum_{n\geq 0} |a_n z|^n \leq M \sum_{n\geq 0} \left|\frac{z}{z_1}\right|^n &\overset{\text{by (4.2)}}{=} M \sum_{n\geq 0} q^n \\ &= \frac{M}{1-q} < \infty, \text{ by Lemma 4.1,}\end{aligned}$$

and $\sum_{n\geq 0} a_n z^n$ converges absolutely for all $z \in \mathbb{D}_{|z_1|}(0)$. Consider now the closed disk $\overline{\mathbb{D}}_\rho(0)$ of radius $\rho < |z_1|$. To prove the uniform convergence of our series we apply the Weierstrass M-test. Note, it follows from (4.3) that for all $z \in \mathbb{D}_\rho(0)$

$$|a_n z^n| \leq M \left|\frac{z}{z_1}\right|^n \leq M \left(\frac{\rho}{|z_1|}\right)^n =: M_n.$$

Since $\frac{\rho}{|z_1|} < 1$, the series $\sum_{n\geq 0} M_n$ converges and therefore, by the Weierstrass M-test, $\sum_{n\geq 0} a_n z^n$ converges uniformly. □

Abel's Theorem has some important corollaries.

Corollary 4.3. *The domain of convergence of a power series is a disk.*

Corollary 4.4. *If at a point z_1 the series $\sum_{n\geq 0} a_n(z-z_0)^n$ diverges it will then diverge on $\mathbb{C} \setminus \mathbb{D}_{|z_1 - z_0|}(z_0)$.*

This fact can easily be proven by contradiction.

Definition 4.5. *The real number R defined by*

$$R = \sup\left\{\rho \;\middle|\; \sum_{n\geq 0} a_n(z-z_0)^n \textit{converges on } \mathbb{D}_\rho(z_0)\right\}$$

is called the radius of convergence of $\sum_{n\geq 0} a_n(z-z_0)^n$.

REMARK 4.6. *Note that R can be 0 or ∞. E.g. R for $\sum_{n\geq 0} n! z_n$ is 0 and R for $\sum_{n\geq 0} \frac{1}{n!} z^n$ is ∞.*

Recall, given a real sequence $\{s_n\}$ we define $\overline{\lim} s_n = \limsup_{n\to\infty} s_n = \lim s_n^*, n \to \infty$ where $s_n^* = \sup\{s_n, s_{n+1}, \ldots\}$. We call this the limit superior of the sequence.

THEOREM 4.7 (The Cauchy-Hadamard Formula). *Let $\sum_{n\geq 0} a_n (z - z_0)^n$ be a power series. Then the radius of convergence can be found by the following formula:*

$$R = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|} \right)^{-1} \tag{4.4}$$

PROOF. Without loss of generality, take $z_0 = 0$ and consider the series $\sum_{n\geq 0} a_n z^n$. Assume first that $0 < R < \infty$. We are going to show that $\sum_{n\geq 0} a_n z^n$ converges for all $z \in \mathbb{D}_R(0)$ and diverges for all $z \in \mathbb{C} \setminus \overline{\mathbb{D}}_R(0)$.

From (4.4) we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R} =: \ell.$$

This implies that for all $\varepsilon > 0$ there exists a N such that $\sqrt[n]{|a_n|} < \ell + \varepsilon$ for all $n \geq N$ and there exist infinitely many $n \geq N$ for which $\sqrt[n]{|a_n|} > \ell - \varepsilon$. We can see what this means more clearly from Figure 2:

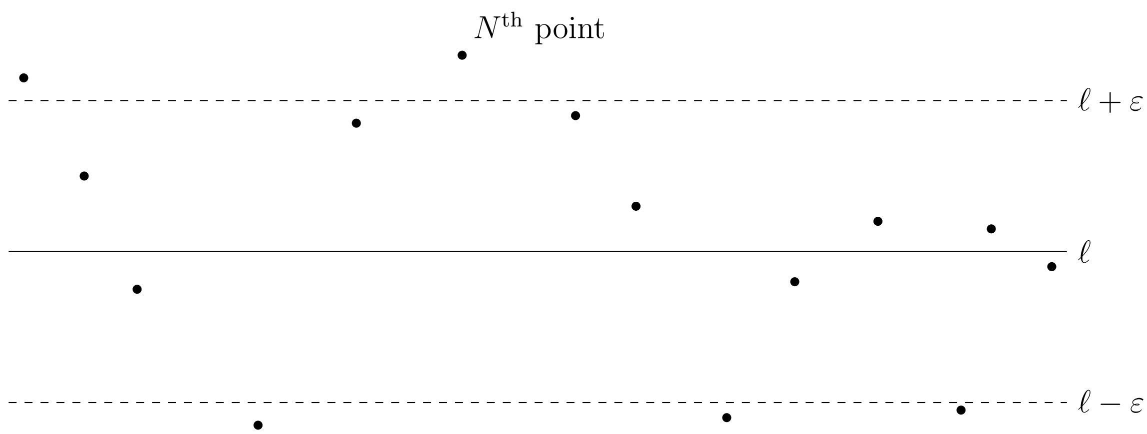


FIGURE 2. Graphical representation of $\limsup \sqrt[n]{|a_n|} = \ell$

Take $z \in \mathbb{D}_R(0)$ and let

$$\varepsilon = \frac{1 - \ell|z|}{2|z|} > 0.$$

Then

$$\begin{aligned}\sqrt[n]{|a_n|}|z| &< (\ell+\varepsilon)|z| = (\ell + \frac{1-\ell|z|}{2|z|})|z| \\ &= \frac{2\ell|z| + 1 - \ell|z|}{2} = \frac{1+\ell|z|}{2} \\ &= q < 1.\end{aligned}$$

Thus $|a_n||z_n|^n < q^n$. But

$$|\sum_{n\geq 0} a_n z^n| \leq \sum_{n\geq 0} |a_n||z|^n < \sum_{n\geq 0} q^n \qquad \text{where } q < 1.$$

Hence by Lemma 4.1 we have that $\sum_{n\geq 0} a_n z^n$ converges for all $z \in \mathbb{D}_R(0)$.

Now select $z \in \mathbb{C} \setminus \overline{\mathbb{D}}_R(0)$ and take

$$\varepsilon = \frac{\ell|z|-1}{|z|} > 0.$$

We have

$$\sqrt[n]{|a_n|}|z| > (\ell-\varepsilon)|z| = (\ell - \frac{\ell|z|-1}{|z|})|z| = 1.$$

Thus for every such z there are infinitely many $n_k \geq N$ such that $|a_{n_k}||z|^{n_k} > 1$ and hence $\sum_{n\geq 0} a_n z^n$ diverges for all z where $|z| > R$. By Definition 4.5 the R defined by (4.4) is the radius of convergence. All that remains is to consider the cases $R = 0$ and $R = \infty$, which are left as an exercise to the reader. $\square$

Example 4.8. *A very important particular case is the geometric series*

$$\sum_{n\geq 0} (z-z_0)^n.$$

Its domain of convergence is $\mathbb{D}_1(z_0)$ and

$$\sum_{n\geq 0} (z-z_0)^n = \frac{1}{1-(z-z_0)}, \quad z \in \mathbb{D}_1(z_0).$$

The proof is left as an exercise.

Exercises

Exercise 4.1 Prove Lemma 4.1.

Exercise 4.2

(1) Show that if in equation (4.4) $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$ then $\sum_{n\geq 0} a_n(z-z_0)^n$ converges for all $z \in \mathbb{C}$ and if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then $\sum_{n\geq 0} a_n(z-z_0)^n$ converges only at $z = z_0$.

(2) Discuss the two examples $\sum_{n\geq 0} n! z_n$ and $\sum_{n\geq 0} \frac{1}{n!} z_n$.

Exercise 4.3 Prove all the statements in Example 4.8.

LECTURE 5

Analytic Functions and the Cauchy-Riemann Conditions

The main objective of this lecture is to introduce the central concept of our course – analytic functions.

Definition 5.1. *Let E be an open set of $\mathbb{C}$ and $f : E \to \mathbb{C}$ and let $z_0 \in E$. If $\dfrac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ converges as $\Delta z \to 0$, then we say that $f(z)$ is differentiable at $z = z_0$, and*

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} =: f'(z_0) \tag{5.1}$$

is called the derivative of $f(z)$ at $z = z_0$.

Remark 5.2. *If z is real, then Definition 5.1 is of course the same as in calculus. However, if z is complex, the existence of the limit (5.1) imposes very restrictive conditions on $\operatorname{Re} f$ and $\operatorname{Im} f$.*

Definition 5.3. *A function $f : E \to \mathbb{C}$ is called differentiable on E if it is differentiable at each point z_0 of E.*

Definition 5.4. *A function $f : E \to \mathbb{C}$ is called continuously differentiable on E if f is differentiable on E and f' is continuous on E.*

Definition 5.5. *A function which is continuously differentiable on E is called analytic on E.*

Note that other names used for analytic functions are regular, or holomorphic.

Remark 5.6. *Note that analyticity of a function is defined on an open set. Hence when we refer to a function f being analytic at a point z_0, it implies that there exists some neighborhood $\mathbb{D}_\delta(z_0)$ where f is differentiable at each point $z \in \mathbb{D}_\delta(z_0)$, and the derivative f' is continuous at each point $z \in \mathbb{D}_\delta(z_0)$.*

Proposition 5.7. *If f, g are analytic on E, then so are:*

(1) $f + g$
(2) fg
(3) f/g *on* $E \setminus \{z \mid g(z) = 0\}$.

The proof is left as an exercise.

Corollary 5.8. *Polynomial functions are analytic on $\mathbb{C}$.*

PROOF. The identity function $\mathbb{I}(z) = z$ and a constant function are clearly analytic on $\mathbb{C}$ and hence by Proposition 5.7 so are $a_k z^k$ for any $k \in \mathbb{N}$ and $a_k \in \mathbb{C}$. Therefore, by Proposition 5.7, the function

$$p_n(z) = \sum_{k=0}^{n} a_k z^k$$

is also analytic on $\mathbb{C}$. □

COROLLARY 5.9. *A rational function* $\dfrac{p_n(z)}{q_m(z)}$ *is analytic on* $\mathbb{C} \setminus \{z \mid q_m(z) = 0\}$.

To see this, just use Corollary 5.8 and Proposition 5.7.

Judging just from Corollaries 5.8 & 5.9, it may seem that there are a lot of analytic functions out there. This is not the case.

EXAMPLE 5.10. *The functions* $\overline{z}$ *and* $|z|$ *are not analytic. Indeed,* $\forall z \in \mathbb{C}$

$$\frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}. \tag{5.2}$$

Now in order for $\overline{z}$ *to be differentiable, the limit of* (5.2) *should be unique, regardless of the path used for* $\Delta z \to 0$. *Yet, if we choose a path along the real line, that is* $\Delta z = \Delta x + i0$, *then*

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

But now, if we consider a path along the imaginary line, that is $\Delta z = 0 + i\Delta y$, *then from* (5.2), *we get*

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

These limits are different, hence (5.2) *diverges for any* z *as* $\Delta z \to 0$.

The proof for $|z|$ is left as an exercise.

REMARK 5.11. *If you pick at random a pair of real-valued functions on* $\mathbb{R}^2$ *such as* $(u(x,y), v(x,y))$, *then the function* $f(z) = u(x,y) + iv(x,y)$ *will most likely be NON-analytic. This is because the set of analytic functions albeit crucially important is a very thin subset of the set of all functions* $u + iv$.

The following theorem gives necessary conditions of analyticity.

THEOREM 5.12 (Cauchy-Riemann Conditions). *If* $f(z) = u(x,y) + iv(x,y)$ *is differentiable at* $z = z_0 = x_0 + iy_0$, *then at* $z = z_0$ *all partial derivatives of* u, v *exist and*

$$\boxed{\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}} \tag{5.3}$$

PROOF. Since f is differentiable at $z = z_0$, then

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f^{'}(z_0) \tag{5.4}$$

does not depend on the way $\Delta z \to 0$. So if we take, as in Example 5.10, $\Delta z = \Delta x$, then (5.4) reads

$$\begin{aligned} f^{'}(z_0) &= \lim_{\Delta x \to 0} \frac{f(z_0 + \Delta x) - f(z_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= (u_x + iv_x)(x_0, y_0). \end{aligned} \tag{5.5}$$

Take in (5.4) $\Delta z = i\Delta y$

$$\begin{aligned} f^{'}(z_0) &= \lim_{\Delta y \to 0} \frac{f(z_0 + i\Delta y) - f(z_0)}{i\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \\ &= -i \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} \\ &= (-iu_y + v_y)(x_0, y_0). \end{aligned} \tag{5.6}$$

Comparing (5.5) and (5.6) yields

$$(u_x + iv_x)(x_0, y_0) = (v_y - iu_y)(x_0, y_0)$$

and (5.3) follows. □

COROLLARY 5.13. *If $f(z)$ is differentiable at $z = z_0$, then*

$$f^{'}(z_0) = (u_x + iv_x)(x_0, y_0) = (v_y - iu_y)(x_0, y_0).$$

This formula is not particularly useful though, as it is almost never a good idea to separate u and v.

COROLLARY 5.14. *If $f : E \to \mathbb{C}$ is analytic on E and u_{xx}, v_{xx}, u_{yy}, v_{yy}, u_{xy}, v_{xy}, u_{yx}, and $v_{yx} \in C(E)$ then u and v satisfy the Laplace equation:*

$$\Delta u = 0 = \Delta v \tag{5.7}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Indeed, differentiating the first equation in (5.3) with respect to y and the second one with respect to x, we obtain:

$$\begin{cases} u_{xx} = v_{yx} \\ u_{yy} = -v_{xy} \end{cases} \tag{5.8}$$

Since $v_{xy} = v_{yx}$, (5.8) implies $\Delta u = 0$. Similarly one proves $\Delta v = 0$.

REMARK 5.15. *Theorem 5.12 and Corollary 5.14 say that* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *of an analytic function* f *must satisfy the Cauchy-Riemann equations* (5.3) *or* (5.7). *The Cauchy-Riemann conditions are necessary conditions for the analyticity of a function.*

Exercises

Exercise 5.1 Prove by definition that:

(1) $f(z) = x^2 - y^2 + 2ixy$ is analytic in $\mathbb{C}$,
(2) $g(z) = x^2 + y^2 + 2ixy$ is not analytic in $\mathbb{C}$. Is $g(z)$ analytic on any open subset of $\mathbb{C}$?

Exercise 5.2 Prove Proposition 5.7. Notice that you can not borrow these statements directly from calculus, but of course you can borrow the idea.

Exercise 5.3 Using the definition of differentiability, show that $|z|$ is not analytic.

Exercise 5.4 Using the Cauchy-Riemann Conditions, show that the following functions are not analytic:

(1) $g(z)$ from Exercise 5.1,
(2) $|z|$.

LECTURE 6

Convergent Power Series are Analytic Functions; e^z

1. Convergent Power Series are Analytic Functions

The title of this section looks like a statement, doesn't it? Yes, I mean it! The statement is so important that I put it in a title.

LEMMA 6.1. *Let $\{a_n\}$ be a monotone decreasing sequence such that $\lim_{n\to\infty} a_n = a$ where $a > 0$. Suppose $\{b_n\}$ is a sequence where $b_n \geq 0$ for all n. Then*

$$\limsup a_n b_n = a \cdot \limsup b_n, \quad n \to \infty.$$

The proof is left as an exercise.

LEMMA 6.2. *Let $\sum_{n\geq 0} a_n z^n$ have a radius of convergence R. Then $\sum_{n\geq 1} n a_n z^n$ also has a radius of convergence R.*

PROOF. Let R_1 be the radius of convergence of $\sum_{n\geq 1} n a_n z^n$. We compute R_1 using the Cauchy-Hadamard formula:

$$R_1^{-1} = \limsup \sqrt[n]{n|a_n|} = \limsup \sqrt[n]{n}\,\sqrt[n]{|a_n|}, \quad n \to \infty. \tag{6.1}$$

Notice that the sequence $\sqrt[n]{n} = \exp \dfrac{\ln n}{n}$ is monotone decreasing (for $n \geq 3$) and

$$\lim \exp \frac{\ln n}{n} = \exp \lim \frac{\ln n}{n} \overset{\text{L'Hospital}}{=} \exp \lim \frac{1}{n} = e^0 = 1 \quad n \to \infty.$$

By Lemma 6.1, equation (6.1) then implies $R_1^{-1} = \limsup \sqrt[n]{|a_n|} = R^{-1}$. □

And now the main result of this lecture.

THEOREM 6.3. *Let $\sum_{n\geq 0} a_n (z-z_0)^n$ converge on $\mathbb{D}_R(z_0)$. Then the function*

$$f(z) = \sum_{n\geq 0} a_n (z-z_0)^n \tag{6.2a}$$

is analytic on $\mathbb{D}_R(z_0)$ and

$$f'(z) = \sum_{n\geq 1} n a_n (z-z_0)^{n-1}. \tag{6.2b}$$

Proof. We use the $\varepsilon/3$ argument. Without loss of generality, we take $z_0 = 0$ and let

$$g(z) = \sum_{n\geq 1} n a_n z^{n-1}. \tag{6.3}$$

By Lemma 6.2, the series (6.3) converges on $\mathbb{D}_R(0)$. Let $z \in \overline{\mathbb{D}}_r(0)$ where $r < R$. Consider

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} - g(z)$$

assuming that $z + \Delta z \in \mathbb{D}_R(z_0)$. Then,

$$\begin{aligned}
\frac{f(z+\Delta z) - f(z)}{\Delta z} - g(z) &= \frac{1}{\Delta z}\Big(\sum_{k\geq 0} a_k (z+\Delta z)^k - \sum_{k\geq 0} a_k z^k\Big) - g(z) \\
&= \underbrace{\frac{S_n(z+\Delta z) - S_n(z)}{\Delta z} - S_n'(z)}_{I} \\
&\quad + \underbrace{S_n'(z) - g(z)}_{II} + \underbrace{\frac{1}{\Delta z}\Big(\sum_{k\geq n+1} a_k (z+\Delta z)^k - \sum_{k\geq n+1} a_k z^k\Big)}_{III}.
\end{aligned}$$

By the triangle inequality we have

$$\left|\frac{f(z+\Delta z) - f(z)}{\Delta z} - g(z)\right| \leq |I| + |II| + |III|. \tag{6.4}$$

We consider $|III|$ first. We have

$$III = \sum_{k\geq n+1} a_k \frac{(z+\Delta z)^k - z^k}{\Delta z}. \tag{6.5}$$

But

$$a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1})$$

and (6.5) can be continued

$$III = \sum_{k\geq n+1} a_k \Big((z+\Delta z)^{k-1} + (z+\Delta z)^{k-2} z + \cdots + z^{k-1}\Big).$$

We now have (since $|z|, |z+\Delta z| < r$)

$$\begin{aligned}
|III| &\leq \sum_{k\geq n+1} |a_k| \Big(|z+\Delta z|^{k-1} + |z+\Delta z|^{k-2}|z| + \cdots + |z|^{k-1}\Big) \\
&\leq \sum_{k\geq n+1} k|a_k| r^{k-1}.
\end{aligned} \tag{6.6}$$

Since by Lemma 6.2 $\displaystyle\sum_{k\geq 1} k a_k z^{k-1}$ converges on $\mathbb{D}_R(0)$, it follows by Abel's Theorem that it is absolutely convergent on $\overline{\mathbb{D}}_r(0)$, $r < R$. Therefore for any $\varepsilon > 0$ there exists

N such that

$$\sum_{k\geq N+1} k|a_k|r^{k-1} < \varepsilon/3.$$

It follows from (6.6) that for all $\varepsilon > 0$ there exists N such that

$$|III| < \varepsilon/3. \tag{6.7}$$

Next we turn our attention to II. Take $n = N$. Then we have

$$\begin{aligned} |II| &= \left|\sum_{k\geq N+1} ka_k z^{k-1}\right| \\ &\leq \sum_{k\geq N+1} k|a_k|r^{k-1} < \varepsilon/3. \end{aligned} \tag{6.8}$$

Finally for $|I|$ we take $n = N$. Since S_n is a polynomial of order N,

$$\frac{S_n(z+\Delta z) - S_n(z)}{\Delta z}$$

approximates $S_n'(z)$ which means that for given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\Delta z| < \delta$ implies that

$$|I| \leq \varepsilon/3. \tag{6.9}$$

Plugging (6.6)-(6.9) into (6.4) we have

$$\left|\frac{f(z+\Delta z) - f(z)}{\Delta z} - g(z)\right| < \varepsilon$$

if $|\Delta z| < \delta$. Therefore $f'(z) = g(z)$, and (6.2b) is proven. Since the series in (6.3) is uniformly convergent on $\overline{\mathbb{D}}_r(0)$ for all $r < R$ we conclude that $g(z)$ is continuous. □

Remark 6.4. *Equation* (6.2*b*) *says that if* $\sum_{n\geq 0} a_n(z-z_0)^n$ *converges on* $\mathbb{D}_R(z_0)$ *then*

$$\frac{d}{dz}\sum_{n\geq 0} a_n(z-z_0)^n = \sum_{n\geq 0} a_n \frac{d}{dz}(z-z_0)^n$$

for all $z \in \mathbb{D}_R(z_0)$.

Corollary 6.5. *If* $f(z) = \sum_{n\geq 0} a_n(z-z_0)^n$ *converges on* $\mathbb{D}_R(z_0)$ *then* $f(z)$ *is* <u>*infinitely differentiable*</u> *on* $\mathbb{D}_R(z_0)$ *and*

$$f^{(k)}(z) = \sum_{n\geq k} n(n-1)\dots(n-k+1)a_n(z-z_0)^{n-k}, \quad k = 1,2,\dots$$

Moreover

$$a_n = \frac{1}{n!} f^{(n)}(z_0).$$

The proof is left as an exercise.

2. Defining the Exponential Function e^z

We now turn our attention to e^z. We are now able to define this object!

DEFINITION 6.6. *Given $z = x + iy \in \mathbb{C}$, the angle θ, $-\pi < \theta \leq \pi$, in the polar representation of the point $(x, y) \in \mathbb{R}^2$ is called the argument of z.*

We assume that $\theta \in (-\pi, \pi]$. Of course we could let $\theta \in [0, 2\pi)$. It is always a good idea to specify the range of θ.

Since $x = r\cos\theta$ and $y = r\sin\theta$ where $r = \sqrt{x^2 + y^2} = |z|$, we have

$$z = |z|(\cos\theta + i\sin\theta) \tag{6.10}$$

which is the trigonometric representation of z. It is common to write $\theta = \arg z$.

PROPOSITION 6.7. *If $z_1, z_2 \in \mathbb{C}$ then*

$$z_1 z_2 = |z_1 z_2|(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

where $\theta_k = \arg z_k, \quad k = 1, 2.$

The proof is left as an exercise.

DEFINITION 6.8. *The function $f(z)$, denoted by e^z or $\exp z$, is called exponential if*

(1)
$$f'(z) = f(z), \tag{6.11}$$

(2) *and*
$$f(0) = 1. \tag{6.12}$$

PROPOSITION 6.9 (de Moivre's Formula). *For all $n \in \mathbb{N}$ and $z \in \mathbb{C}$*

$$z^n = |z|^n(\cos(n\theta) + i\sin(n\theta)) \tag{6.13}$$

The proof is left as an exercise.

THEOREM 6.10.

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}.$$

The proof is left as an exercise.

Exercises

Exercise 6.1 Prove Lemma 6.1.

Exercise 6.2 Prove that $(z^n)' = nz^{n-1}$, $n = 0, 1, \ldots$

Exercise 6.3 Prove Corollary 6.5.

Exercise 6.4 Prove Proposition 6.7.

Exercise 6.5 Prove Proposition 6.9.

Exercise 6.6 Prove Theorem 6.10.

LECTURE 7

The Chain Rule; More on e^z, $\sin z$, and $\cos z$

We continue to study e^z. We also introduce basic trigonometric functions. But one long overdue statement is first.

1. The Chain Rule

LEMMA 7.1. *f(z) is differentiable on E iff for all $z, z+\Delta z \in E$*

$$f(z+\Delta z) - f(z) = A(z)\Delta z + \varepsilon(z, \Delta z)$$

where $A(z) = f'(z)$ and $\varepsilon(z, \Delta z)$ is bounded for every $z \in E$ and

$$\lim_{\Delta z \to 0} \frac{\varepsilon(z, \Delta z)}{\Delta z} = 0.$$

The proof is left as an exercise.

THEOREM 7.2 (The Chain Rule). *If functions $g : E \to G$ and $f : G \to \mathbb{C}$ are analytic on E and G respectively, then $(f \circ g)(z) := f(g(z))$ is also analytic on E. In addition,*

$$(f \circ g)'(z) = f'(g(z)) \cdot g'(z) \tag{7.1}$$

for all $z \in E$.

PROOF. Let $z_0, z_0 + \Delta z \in E$ and find $w_0, w_0 + \Delta w \in G$ such that $w_0 = g(z_0)$ and $w_0 + \Delta w = g(z_0 + \Delta z)$.

By Lemma 7.1, since f is differentiable at $w = w_0$

$$f(w_0 + \Delta w) - f(w_0) = f'(w_0)\Delta w + \varepsilon_1(w_0, \Delta w).$$

By the same lemma,

$$\lim_{\Delta w \to 0} \frac{\varepsilon_1(w_0, \Delta w)}{\Delta w} = 0 \tag{7.2}$$

and

$$\Delta w = g(z_0 + \Delta z) - g(z_0) = g'(z_0)\Delta z + \varepsilon_2(z_0, \Delta z). \tag{7.3}$$

Since g is differentiable at $z = z_0$, combining (7.2) and (7.3) we get

$$\begin{aligned} f(g(z_0{+}\Delta z)) &- f(g(z_0)) \\ &= f'(g(z_0))\left(g'(z_0)\Delta z + \varepsilon_2(z_0, \Delta z)\right) + \left(\varepsilon_1(g(z_0), \Delta w)\right) \\ &= f'(g(z_0))g'(z_0)\Delta z + f'(g(z_0))\varepsilon_2(z_0, \Delta z) + \varepsilon_1(g(z_0), \Delta w). \end{aligned}$$

The last two terms of this equation shall define $\varepsilon(z_0, \Delta z)$.

Differentiability of $f \circ g$ and equation (7.1) will be proven if we show

$$\lim_{\Delta z \to 0} \frac{\varepsilon(z_0, \Delta z)}{\Delta z} = 0.$$

We know $\Delta w = g(z_0 + \Delta z) - g(z_0)$. Thus

$$\frac{\varepsilon(z_0, z)}{\Delta z} = f'(g(z_0)) \frac{\varepsilon_2(z_0, \Delta z)}{\Delta z} + \frac{\varepsilon_1(g(z_0), \Delta w)}{\Delta z}$$

the first term of which goes to zero as Δz goes to zero because g is analytic and f' is bounded.

For the second term we need to prove that for any sequence $\Delta z_n \to 0$,

$$\frac{\varepsilon_1(g(z_0), g(z_0 + \Delta z_n) - g(z_0))}{\Delta z_n} \to 0.$$

Consider the decomposition:

$$\{\Delta z_n\} = \{\Delta z_n'\} \cup \{\Delta z_n''\}$$

where

$$\begin{aligned} &\{\Delta z_n'\} = \{\Delta z_n' \,|\, g(z_0 + \Delta z_n') \neq g(z_0)\} \\ \text{and} \qquad &\{\Delta z_n''\} = \{\Delta z_n'' \,|\, g(z_0 + \Delta z_n'') = g(z_0)\}. \end{aligned}$$

Now, $\Delta w_n' := g(z_0 + \Delta z_n') - g(z_0)$. Then

$$\begin{aligned} \lim_{\Delta z_n' \to 0} \frac{\varepsilon_1(w_0, \Delta w_n')}{\Delta z_n'} &= \lim_{\Delta z_n' \to 0} \frac{\varepsilon_1(w_0, \Delta w_n')}{\Delta w_n'} \cdot \frac{\Delta w_n'}{\Delta z_n'} \\ &= \underbrace{\lim_{\Delta z_n' \to 0} \frac{\varepsilon_1(w_0, \Delta w_n')}{\Delta w_n'}}_{=0 \text{ by } (7.3)} \cdot \underbrace{\lim_{\Delta z_n' \to 0} \frac{g(z_0 + \Delta z_n') - g(z_0)}{\Delta z_n'}}_{g'(z_0)} \\ &= 0. \end{aligned}$$

One does a similar computation for the subsequence $\{\Delta z_n''\}$.

$$\lim_{\Delta z_n'' \to 0} \frac{\varepsilon_1(w_0, g(z_0 + \Delta z_n'') - g(z_0))}{\Delta z_n''} = \lim_{\Delta z_n'' \to 0} \frac{\varepsilon_1(w_0, 0)}{\Delta z_n''} = 0$$

since by (7.2), $\varepsilon_1(z_0, 0) = 0$.

Thus (7.1) is proved. Since Δz_n is arbitrary, both factors on the right hand side of (7.1) are continuous and hence so is $(f \circ g)'$. Thus $f \circ g$ is analytic on E.

$\square$

Corollary 7.3. $\dfrac{1}{f(z)}$ *is analytic on* $E \setminus \{z \,|\, f(z) = 0\}$ *if* $f(z)$ *is analytic on* E.

2. Exponential and Trigonometric Functions

PROPOSITION 7.4 (The main proposition regarding e^z). *For all $z_1, z_2 \in \mathbb{C}$, we have*

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}. \tag{7.4}$$

PROOF. Without loss of generality, let $z_1 = z$ and $z_2 = z_0 - z_1$ where z_0 is a fixed number.

Consider $g(z) := e^z \cdot e^{z_0 - z}$. By the product and chain rules,

$$g'(z) = e^z \cdot e^{z_0-z} - e^z \cdot e^{z_0-z} = 0.$$

By Corollary 6.5, $g(z) = c$, a constant, for all $z \in \mathbb{C}$. Hence $c = g(0) = e^0 \cdot e^{z_0-0}$. Thus $e^z \cdot e^{z_0-z} = e^{z_0}$. Going back to z_1, z_2 yields (7.4). □

THEOREM 7.5 (Euler's Formula).

$$e^{ix} = \cos x + i \sin x, \qquad x \in \mathbb{R}$$

PROOF. Recall

$$e^{ix} = \sum_{n\geq 0} \frac{(ix)^n}{n!}. \tag{7.5}$$

But $i^{2k} = (-1)^k$ and $i^{2k+1} = (-1)^k i$, hence

$$\begin{aligned}(7.5) &= \sum_{k\geq 0} \frac{(ix)^{2k}}{(2k)!} + \sum_{k\geq 0} \frac{(ix)^{2k+1}}{(2k+1)!} \\ &= \sum_{k\geq 0} \frac{(-1)^k (x)^{2k}}{(2k)!} + i \sum_{k\geq 0} \frac{(-1)^k (x)^{2k+1}}{(2k+1)!} \\ &= \cos x + i \sin x.\end{aligned}$$

□

PROPOSITION 7.6. *The following hold for the complex exponential:*

(1) $e^z = e^{\operatorname{Re} z} (\cos(\operatorname{Im} z) + i \sin(\operatorname{Im} z))$.
(2) $|e^z| = e^{\operatorname{Re} z}$.
(3) e^z *is periodic with minimal period* $2\pi i$.
(4) $\overline{e^z} = e^{\overline{z}}$.
(5) $\operatorname{Ran}(e^z) = \mathbb{C} \setminus \{0\}$.

The proof is left as an exercise.

COROLLARY 7.7 (The exponential form of a complex number).

$$z = |z| e^{i \arg z}.$$

DEFINITION 7.8. *For any $z \in \mathbb{C}$, we define*

$$\sin z := \sum_{n\geq 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

and

$$\cos z := \sum_{n\geq 0} \frac{(-1)^n z^{2n}}{(2n)!}.$$

REMARK 7.9. *When we restrict z to the real axis, Definition 7.8 agrees with our usual definition of sine and cosine on the real numbers.*

PROPOSITION 7.10.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

The proof is left as an exercise.

PROPOSITION 7.11. *Properties of the the complex trigonometric functions:*

(1) $e^{iz} = \cos z + i \sin z$.
(2) $\sin^2 z + \cos^2 z = 1$.
(3) $\sin 2z = 2 \sin z \cos z$.
(4) $\mathrm{Ran}(\sin) = \mathrm{Ran}(\cos) = \mathbb{C}$.

The proof is left as an exercise.

REMARK 7.12. *When we define the exponential and trigonometric functions on the complex numbers, these functions pick up a few new properties. In particular, equations such as*

$$e^z = 1$$
$$\sin z = 2$$
$$\cos z = 2.$$

now have (infinitely many) solutions!

Exercises

Exercise 7.1 Prove Lemma 7.1.

Exercise 7.2 Prove Proposition 7.6.

Exercise 7.3 Prove Proposition 7.10 and use it to prove Proposition 7.11.

LECTURE 8

N^{th} Roots; Univalent Functions

With Euler's formula in hand we are now able to define radicals of complex numbers.

Definition 8.1. *Let $w \neq 0$, $n \in \mathbb{N}$, a number z is called an* $\underline{n^{th} \text{ root}}$ *of w if $z^n = w$.*

Proposition 8.2 (Formula for the n^{th} roots). *Let $w = |w|e^{i\theta} \neq 0$ and $n \in \mathbb{N}$, then w has n distinct n^{th} roots $z_1, z_2, ..., z_n$ given by*

$$z_k = \sqrt[n]{|w|} \exp i \frac{\theta + 2\pi(k-1)}{n}, \ \ k = 1, 2, ..., n \tag{8.1}$$

Proof. Consider

$$z_1 = \sqrt[n]{|w|} e^{i\frac{\theta}{n}}.$$

By de Moivre's formula

$$z_1^n = |w| e^{i\theta} = w$$

and hence z_1 is a n^{th} root of w.

Define now

$$z_k := z_1 e^{i\frac{2\pi(k-1)}{n}}, \ \ k \in \mathbb{N} \tag{8.2}$$

One has

$$z_k^n = \underbrace{z_1^n}_{=w} \underbrace{e^{2\pi i(k-1)}}_{=1} = w$$

Thus so-defined sequence $\{z_k\}$ gives n^{th} roots. It follows from (8.2) that

$$z_k = z_{k-1} e^{i\frac{2\pi}{n}}$$

which implies (the details should be verified in Exercise 8.1) that all $z_1, z_2, ..., z_n$ are distinct but $z_{n+1} = z_1$, $z_{n+2} = z_2$, ... , $z_{2n} = z_n$. $\square$

Example 8.3. *Find the sixth roots of unity, i.e., find all solutions to $z^6 = 1$.*

By (8.1)

$$z_k = \exp i \frac{2\pi(k-1)}{6}, \ \ k = 1, 2, ..., 6$$

which gives $z_1 = 1$, $z_2 = 1/2 + i\sqrt{3}/2$, $z_3 = -1/2 + i\sqrt{3}/2$, $z_4 = -1$.

To find z_5 and z_6 notice that equation $z^6 = 1$ is equivalent to $\overline{z}^6 = 1$ and hence $\overline{z_2}, \overline{z_3}$ are also solutions to $z^6 = 1$. Thus $z_5 = \overline{z_3} = -1/2 - i\sqrt{3}/2$, and $z_6 = \overline{z_2} = 1/2 - i\sqrt{3}/2$.

Figure 1 shows the six sixth roots of unity on the complex plane.

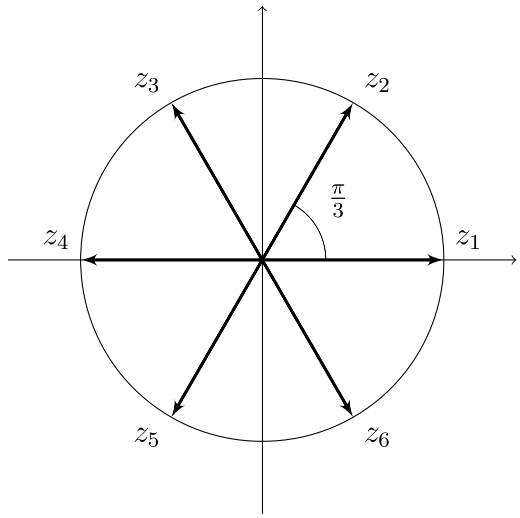


FIGURE 1. The sixth roots of the unity

REMARK 8.4. *Proposition 8.2 says that the equation*

$$z^{2008} = 1$$

has 2008 distinct solutions and if you mistakenly think that $z = \pm 1$ *then you lose 2006 solutions.*

What we have discussed in this lecture so far suggests that defining the function $\sqrt[n]{z}$ will require some effort.

DEFINITION 8.5. *An analytic function* $f : E \to \mathbb{C}$ *is called* <u>*univalent*</u> *on* E *if*

$$f(z_1) = f(z_2) \Rightarrow z_1 = z_2.$$

Note that generic complex valued functions don't like to be univalent. E.g. $f(z) = |z|$ maps any circle $|z| = r$ onto one point $\{r\} \in \mathbb{R}$, and hence $|z|$ is very non-univalent.

EXAMPLE 8.6. *The linear function* $f(z) = az + b$ *is univalent (if* $a \neq 0$*) on* $\mathbb{C}$*. The function* $\phi(z) = z/a - b/a$ *satisfies*

$$f \circ \phi = \phi \circ f = \mathbb{I}$$

and hence ϕ *can be viewed as the inverse of* f*.*

EXAMPLE 8.7. *The function* $f(z) = 1/z$ *is univalent on* $\mathbb{C} \setminus \{0\}$ *and its inverse is* $\phi(z) = 1/z$*.*

EXAMPLE 8.8. *The function* $f(z) = z^2$ *is not univalent on* $\mathbb{C}$*. But as in the real valued case we can properly restrict* z^2 *to make it univalent. To do so we consider* z^2 *as a* <u>*mapping*</u> *on* $\mathbb{C}$*.*

We introduce a <u>*sector*</u>*:*

$$S(\alpha, \beta) := \{z | \ \alpha < \arg z < \beta\}.$$

Let $z \in S(\alpha, \beta)$. We have $z = |z|e^{i\theta}, \alpha < \theta < \beta$, and hence

$$z^2 = |z|^2 e^{2i\theta} \in S(2\alpha, 2\beta).$$

In particular, if $\alpha = 0$ and $\beta = 2\pi$ then

$$z^2 : S(0, 2\pi) \to S(0, 4\pi).$$

That is, loosely speaking, z^2 maps $\mathbb{C}$ onto two copies of $\mathbb{C}$.
However if we take $\alpha = 0$ and $\beta = \pi$ then

$$z^2 : S(0, \pi) \to S(0, 2\pi)$$

which is a univalent function. See Figure 2 for illustration.

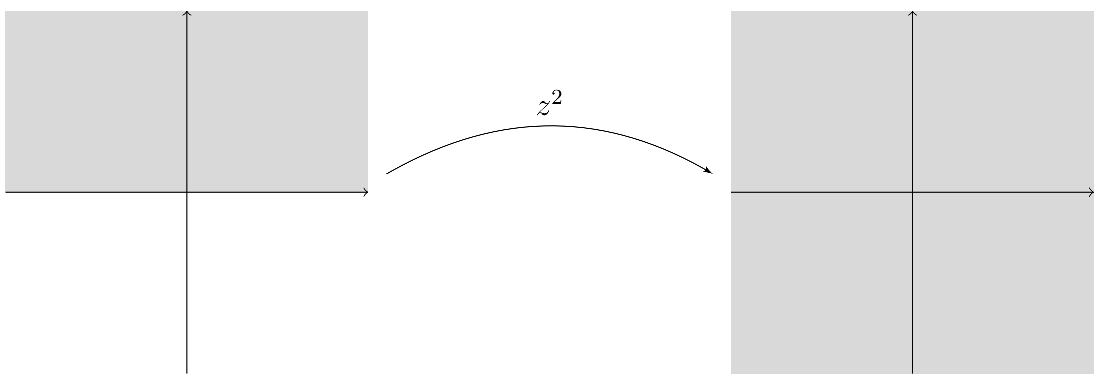


FIGURE 2. Transformation of the upper half plane under z^2

Exercises

Exercise 8.1 Let $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ be the n^{th} roots of unity. Pick a fixed $\omega \in \Omega$. Show

(1) $\omega\omega_k \in \Omega$.
(2) $\omega\omega_k \neq \omega\omega_m$ if $k \neq m$, i.e., $\{\omega\omega_k\}_{k=1}^n = \Omega$.
(3) $\displaystyle\sum_{k=1}^{n} \omega_k = 0$.
(4) $\displaystyle\sum_{k=0}^{n-1} \omega^k = 0$ for any $\omega \neq 1$.

Exercise 8.2 Describe a linear function as a mapping, i.e., describe what geometric transformation $f(z) = az + b$ represents on $\mathbb{C}$. Find the images $f(\Omega)$ of Ω for the following sets:

(1) $\Omega = \{z| \ \arg z = \theta_0\}$, where θ_0 is fixed. (A ray)
(2) $\Omega = \{z| \ |z| = r_0\}$, where r_0 is fixed. (A circle)

Exercise 8.3 Let $f(z) = 1/z$. Find the images $f(\Omega)$ of the following sets:

(1) Ω is a straight line passing through 0.
(2) $\Omega = \{z| \ |z| = r_0\}$, where r_0 is fixed.

Describe the geometrical transformation the function $f(z)$ does to point z in Figure 3.

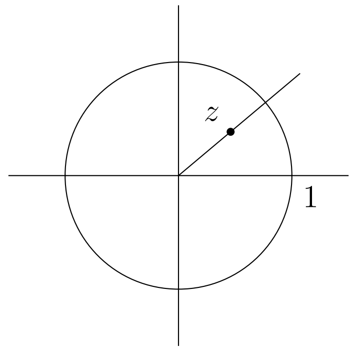


FIGURE 3. Transformation of z under $1/z$?

LECTURE 9

z^n, $\sqrt[n]{z}$, e^z, $\log z$ and all that

In this lecture we continue to study elementary functions of a single complex variable as mappings.

1. The Function z^n

Let $z = re^{i\theta}$. Then

$$w := z^n = r^n e^{in\theta}$$

and hence z^n maps a sector $S(\alpha, \beta)$ onto the sector $S(n\alpha, n\beta)$. In particular, if $\alpha = 0$ and $\beta = 2\pi$, then z^n maps $\mathbb{C} = S[0, 2\pi)$ onto n copies of $\mathbb{C}$. In other words, z^n is an n-valent function.

Now restrict z^n to $S\left(0, \frac{2\pi}{n}\right)$. Then $w \in S(0, 2\pi)$. Moreover, every ray

$$R_\theta := \left\{ z \,|\, z = re^{i\theta}, r \in \mathbb{R}_+ \right\}, \quad \text{where } \mathbb{R}_+ = [0, \infty),$$

in $S\left(0, \frac{2\pi}{n}\right)$ is mapped onto the ray $R_{n\theta}$.

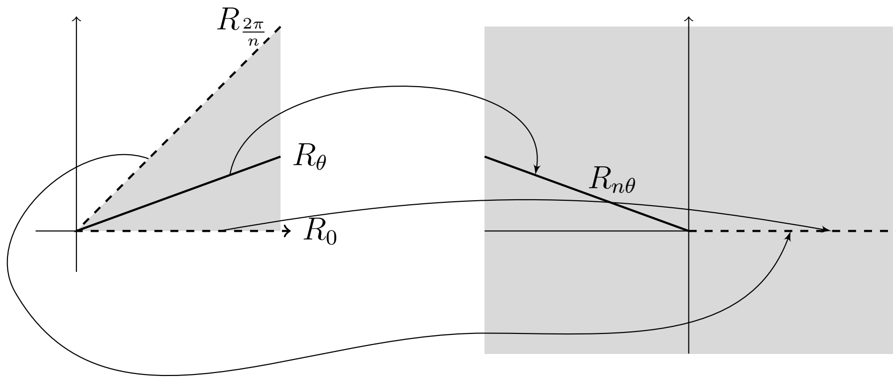


FIGURE 1. Mapping of a sector under z^n

Observe that $z^n : S\left(0, \frac{2\pi}{n}\right) \to S(0, 2\pi)$ is univalent and we may now introduce the inverse of z^n.

2. The Function $\sqrt[n]{z}$

As we have seen in Lecture 8, the equation $z^n = w$ has n solutions $\{z_1, z_2, \ldots, z_n\}$. In order to define $\sqrt[n]{w}$ we have to decide which of these solutions we want to pick up. It is a personal choice, but common sense suggests that we choose z_1. Doing so allows $\sqrt[n]{1} = 1$, and has the nice property that our complex n^{th} root function will correspond to the usual positive n^{th} root function on the real number line (or on $\mathbb{R}_+$ if n is even).

DEFINITION 9.1. *Let $z = |z|e^{i\arg z} \neq 0$ where $\arg z \in [0, 2\pi)$. The principal branch of $\sqrt[n]{z}$ is defined by*

$$\sqrt[n]{z} = \sqrt[n]{|z|}e^{\frac{i\arg z}{n}}. \tag{9.1}$$

REMARK 9.2. *$\sqrt[n]{z}$ can also be denoted as $z^{\frac{1}{n}}$. In some books, they distinguish between these two representations, using one to denote the multi-valued function and the other to denote the principal branch. We choose not to do so in this course. Also, note that the principal n^{th} root of z defined by (9.1) is z_1 in the n^{th} root formula (given in Lecture 8). If occasionally we choose a different branch of $\sqrt[n]{z}$ we must specify how we define it.*

Let us now look at (9.1) as a function of z. Formula (9.1) defines a one-to-one function $\sqrt[n]{z}$,

$$\sqrt[n]{z} : S(0, 2\pi) \to S\left(0, 2\pi/n\right). \tag{9.2}$$

PROPOSITION 9.3. *The function $\sqrt[n]{z}$ defined by (9.1) and (9.2) is analytic on $\mathbb{C}\setminus\mathbb{R}_+$ but not analytic on $\mathbb{C}$.*

The proof is left as an exercise.

3. The Function e^z

Consider a strip

$$K_0 = \{z : 0 < \operatorname{Im} z < 2\pi\}.$$

Let $z = x + iy \in K_0$. Then $e^z = e^x e^{iy}$ maps K_0 onto $\mathbb{C} \setminus \mathbb{R}_+$. Moreover, every vertical segment

$$k_x = \{z = x + iy \,:\, y \in (0, 2\pi)\}$$

is mapped onto the circle $C_{e^x}(0) \setminus \mathbb{R}_+$, where

$$C_r(z_0) := \{z \,:\, |z - z_0| = r\}.$$

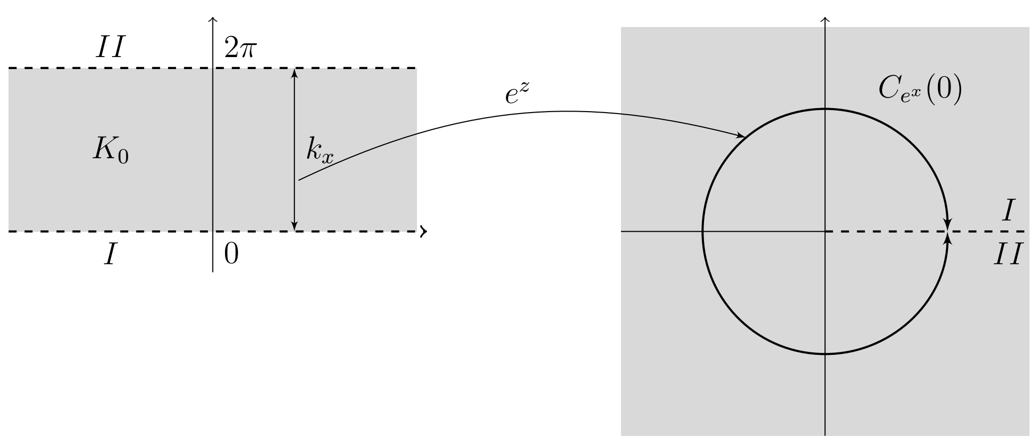


FIGURE 2. Mapping of a strip under e^z

Observe that e^z is univalent on K_0. Let

$$K_{2\pi} := \{z \,:\, 2\pi < \operatorname{Im} z < 4\pi\}.$$

It is clear that e^z maps $K_{2\pi}$ onto $\mathbb{C} \setminus \mathbb{R}_+$. Similarly, for all $n \in \mathbb{Z}$,

$$\exp K_{2\pi n} = \mathbb{C} \setminus \mathbb{R}_+,$$

which means that e^z is an infinitely-valent (infinitely valued) function.

Remark 9.4. *We keep our strips open for a purpose which will be clear in the next section. Of course, e^z maps $\mathbb{R}$ onto $\mathbb{R}_+ \setminus \{0\}$ and maps*

$$\mathbb{R} + 2i\pi := \{z \,|\, z = x + iy, x \in \mathbb{R}, y = 2\pi\}$$

onto $\mathbb{R}_+ \setminus \{0\}$.

4. The Function $\log z$

Since the mapping

$$e^z : K_0 \to \mathbb{C} \setminus \mathbb{R}_+$$

is one-to-one, we can formally define the complex logarithm.

Definition 9.5. *Let $z = |z|e^{i \arg z} \neq 0$ where $\arg z \in [0, 2\pi)$. The principal branch of $\log z$ is defined by*

$$\log z = \log|z| + i \arg z \tag{9.3}$$

where $\log|z|$ is the usual natural logarithm of the positive real number $|z|$.

Remark 9.6. *In the literature, the complex natural logarithm is sometimes denoted* ln *or* Log. *In complex analysis* log, ln, *and* Log *all refer to the logarithm base e, as opposed to real analysis where* log *often denotes the logarithm base* 10.

Let us now look at (9.3) as a function of z:

$$\log z : \mathbb{C} \setminus \mathbb{R}_+ \to K_0. \tag{9.4}$$

Proposition 9.7. *The function $\log z$ defined by* (9.3) *and* (9.4) *is the inverse of*

$$e^z : K_0 \to \mathbb{C} \setminus \mathbb{R}_+.$$

The proof is left as an exercise.

Proposition 9.8. *The complex logarithm function $\log z$ is analytic on $\mathbb{C} \setminus \mathbb{R}_+$, but is not analytic on $\mathbb{C}$. Furthermore*

$$(\log z)' = \frac{1}{z} \quad , \quad z \in \mathbb{C} \setminus \mathbb{R}_+.$$

The proof is left as an exercise.

Exercises

Exercise 9.1 Prove Proposition 9.3.

Exercise 9.2 Prove Proposition 9.7.

Exercise 9.3 Prove Proposition 9.8.

LECTURE 10

$\log z$ continued; $\sin z$, and $\cos z$

1. $\log z$ continued

In this lecture we continue studying elementary functions. We start with an important remark.

REMARK 10.1. *As we have seen, e^z retains all of its original properties and on top of that, e^z also gains some new ones. (eg.* $\mathrm{Ran}(e^z) = \mathbb{C} \setminus 0$ *not just* $\mathbb{R}_+$, *and e^z is now periodic). On the contrary* $\log z$ *loses some of its common properties. E.g. the property* $\log(z_1 z_2) = \log z_1 + \log z_2$, *alas, no longer holds in general. So watch out!*

REMARK 10.2. *Our definition of the principal branch of* $\log z$ *(Definition 9.5) is not the only one. All depends on a particular situation. Another common way to choose the principal branch is to let* $z = |z|e^{i \arg z}$ *where* $\arg z \in [-\pi, \pi)$, *(not* $[0, 2\pi)$*). Then*

$$\log z := \log |z| + i \arg z. \tag{10.1}$$

I.e. this $\log$ *is defined on* $\mathbb{C} \setminus \mathbb{R}_-$.

The log defined by (10.1) has the nice property $\log \bar{z} = \overline{\log z}$ where as the one we defined before did not.

Before we are done with logarithms, let us state and prove an important proposition.

PROPOSITION 10.3. *Suppose* $f : E \to \mathbb{C}$ *and* $g : f(E) \to \mathbb{C}$ *are continuous and*

$$g(f(z)) = z. \tag{10.2}$$

Then,

(1) *if g is differentiable and $g'(z) \neq 0$, then f is differentiable and*

$$f'(z) = (g'(f(z)))^{-1} \tag{10.3}$$

(2) *if g is analytic and $g'(f(z)) \neq 0$, then f is analytic.*

PROOF. Assume that f is continuous and let $z, z + \Delta z \in E$. Then from (10.2) we see that:

$$g(f(z)) = z, \;\; g(f(z + \Delta z)) = z + \Delta z \tag{10.4}$$

$$\begin{aligned}
&\Rightarrow f(z) \neq f(z + \Delta z) \text{ if } \Delta z \neq 0 \\
&\Rightarrow \Delta f(z) := f(z + \Delta z) - f(z) \neq 0 \\
&\Rightarrow f(z + \Delta z) = f(z) + \Delta f(z).
\end{aligned}$$

Consider $\dfrac{g(f(z+\Delta z))-g(f(z))}{\Delta z}$. We have by equation (10.4)

$$1=\frac{g(f(z+\Delta z))-g(f(z))}{\Delta z}=\underbrace{\frac{g(f(z)+\Delta f(z))-g(f(z))}{\Delta f(z)}}_{\to g'(f(z)) \text{ as } \Delta f(z)\to 0}\cdot\frac{\Delta f(z)}{\Delta z}.$$

Then by taking the limit as Δz goes to zero we see that

$$\begin{aligned} & 1=\lim_{\Delta z\to 0}\frac{g(f(z)+\Delta f(z))-g(f(z))}{\Delta f(z)}\cdot\frac{\Delta f(z)}{\Delta z} \\ \Rightarrow\quad & 1=g'(f(z))\cdot\lim_{\Delta z\to 0}\frac{\Delta f(z)}{\Delta z}. \end{aligned} \tag{10.5}$$

(We have used $\Delta f(z)\to 0$ as $\Delta z\to 0$ due to continuity of f.) It now follows from (10.5) that the limit on the right hand side exists and (10.3) follows. Statement (1) is proven.

If g is analytic then g' is continuous and so f' is continuous. Then statement (2) is also proven because f' is continuous which implies that f is analytic.

□

Corollary 10.4. *The function* $\log z$ *is analytic on* $\mathbb{C}\setminus\mathbb{R}_+$ *and*

$$(\log z)'=\frac{1}{z}. \tag{10.6}$$

Proof. Using Proposition 10.3 $f(z)=\log z$, $E=\mathbb{C}\setminus\mathbb{R}_+$ and $g(z)=e^z$. Since

$$\exp\log z=z \text{ for all } z\in\mathbb{C}\setminus\mathbb{R}_+$$

(10.2) holds and the conditions of Proposition 10.3 are satisfied since e^z is analytic. Thus, $\log z$ is also analytic and by (10.3)

$$\begin{aligned}(\log z)'&=\left(\frac{d}{dw}e^w\,\big|_{w=\log z}\right)^{-1}\\ &=(e^{\log z})^{-1}=\frac{1}{z}.\end{aligned}$$

□

Remark 10.5. *Did you enjoy proving it by definition? Would Proposition 10.3 have helped prove Exercise 9.3?*

2. The Functions $\sin z$, $\cos z$, and $\tan z$

We restrict ourselves to $\sin z$ only. Then by definition,

$$\begin{aligned}\sin z &= \frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i} = \frac{e^{-y}e^{ix}-e^{y}e^{-ix}}{2i}\\ &= \frac{1}{2i}(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x))\\ &= \frac{1}{2i}(e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x - ie^{y}\sin x)\\ &= \frac{e^{y}\sin x + e^{-y}\sin x}{2} + i\frac{e^{y}\cos x - e^{-y}\cos x}{2}\\ &= \sin x\cosh y + i\cos x\sinh y.\end{aligned}$$

I told you once that it is almost never a good idea to separate the real and imaginary parts of the function. This is a rare case of when it is needed. So we get

$$\sin(x+iy) = \underbrace{\sin x\cosh y}_{u} + i\underbrace{\cos x\sinh y}_{v}.$$

We have

$$\sin x = \frac{u}{\cosh y}, \qquad \cos x = \frac{v}{\sinh y}$$

and hence

$$1 = \sin^2 x + \cos^2 x = \left(\frac{u}{\cosh y}\right)^2 + \left(\frac{v}{\sinh y}\right)^2. \tag{10.7}$$

It follows from (10.7) that $\sin z$ maps a segment $\{z|z = x+iy_0, x\in[-\frac{\pi}{2},\frac{\pi}{2}]\}$, where $y_0 \geq 0$ is fixed, onto the upper semiellipse

$$\left(\frac{u}{\cosh y_0}\right)^2 + \left(\frac{v}{\sinh y_0}\right)^2 = 1.$$

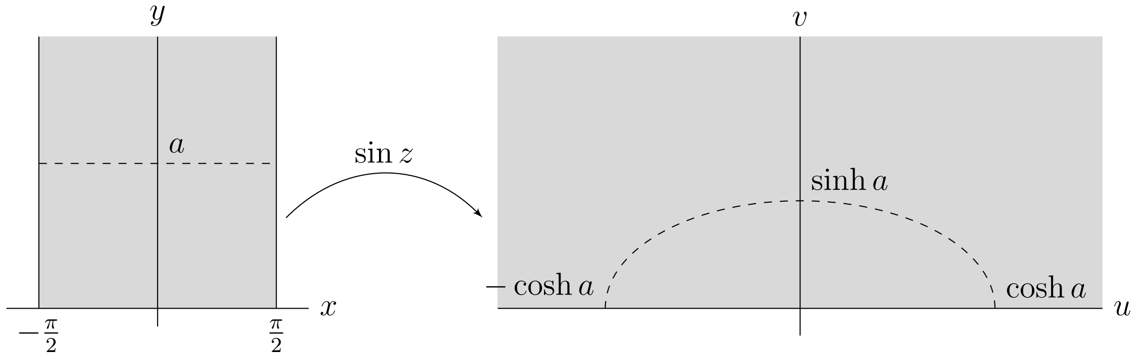


FIGURE 1. Mapping of $\sin z$ for $z = x + ia$ where $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $a \geq 0$

Similarly, one can see

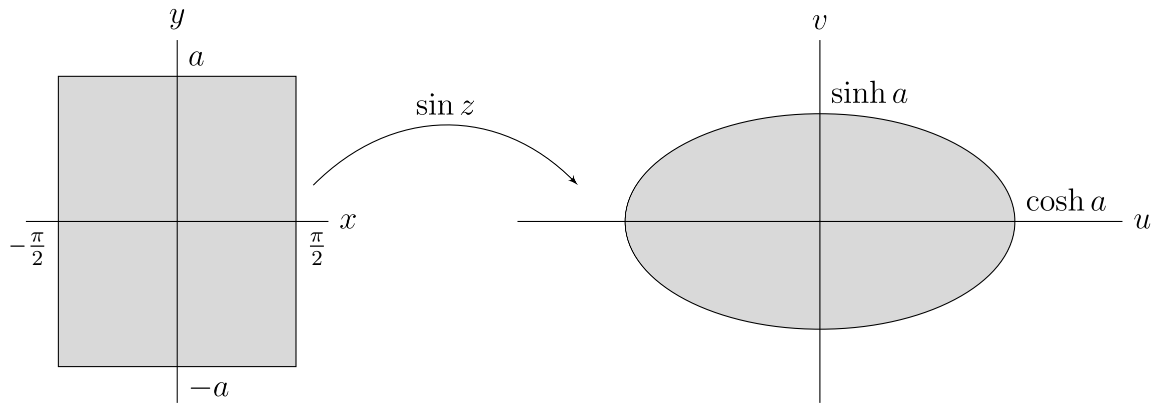


FIGURE 2. Mapping of $\sin z$ for $z = x + iy$ where $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $y \in [-a, a]$

Exercises

Exercise 10.1 Let $z_1, z_2, ..., z_n \in \mathbb{C}^+ := \{z | \operatorname{Im} z > 0\}$. Show that if $z_1 \cdot z_2 \cdot \ldots \cdot z_k \in \mathbb{C}^+$ for all $k \leq n$ then,

$$\log \prod_{k=1}^{n} z_k = \sum_{k=1}^{n} \log z_k, \tag{10.8}$$

where the log is defined on $\mathbb{C} \backslash \mathbb{R}_+$. Give a counterexample to equation (10.8) if we remove the restriction on $z_1, z_2, ..., z_n$. Note, in this respect, that in general

$$\log z^n \neq n \log z.$$

Exercise 10.2 Let $\log z$ be defined on $\mathbb{C} \setminus \mathbb{R}_+$. Come up with reasonable conditional statements regarding the basic properties of $\log z$.
E.g. $\log z_1 z_2 = \log z_1 + \log z_2$, if blah, blah, blah.

Exercise 10.3 Treat $\cos z$ in a way similar to what we did with $\sin z$.

LECTURE 11

Conformal Maps

We continue to study analytic functions as mappings. Let us understand the geometric meaning of the derivative. Since derivatives are now complex-valued, the calculus interpretation of the derivative no longer makes sense. But we are going to find a new one.

1. Geometric Interpretation of the Derivative

Let $f : E \to \mathbb{C}$ be analytic and $f'(z_0) \neq 0$ where $z_0 \in E$ is a fixed point and $w_0 := f(z_0)$. Write $f'(z_0)$ in exponential form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w(z_0)}{\Delta z} = Ae^{i\alpha} \tag{11.1}$$

where as before $\Delta w(z_0) = f(z_0 + \Delta z) - f(z_0)$. The limit in (11.1) doesn't depend on how $\Delta z \to 0$. So we take two different paths γ_1, γ_2 non-tangential at z_0 as shown on Figure 1.

Let $\Gamma_1 = f(\gamma_1), \Gamma_2 = f(\gamma_2)$ be the images of γ_1, γ_2 under f.

From (11.1),

$$\alpha = \arg f'(z_0) = \lim_{\Delta z \to 0} \arg \Delta w - \lim_{\Delta z \to 0} \arg \Delta z. \tag{11.2}$$

Since α is independent of the way $\Delta z \to 0$, (11.2) yields

$$\underbrace{\lim_{\Delta z \to 0,\ z \in \gamma_1} \arg \Delta w}_{\phi_1} - \underbrace{\lim_{\Delta z \to 0,\ z \in \gamma_1} \arg \Delta z}_{\varphi_1} = \underbrace{\lim_{\Delta z \to 0,\ z \in \gamma_2} \arg \Delta w}_{\phi_2} - \underbrace{\lim_{\Delta z \to 0,\ z \in \gamma_2} \arg \Delta z}_{\varphi_2}$$

$$\begin{aligned} &\Rightarrow \quad \phi_1 - \varphi_1 = \phi_2 - \varphi_2 \\ &\Rightarrow \quad \underbrace{\phi_2 - \phi_1}_{\Delta\phi} = \underbrace{\varphi_2 - \varphi_2}_{\Delta\varphi} \\ &\Rightarrow \quad \Delta\phi = \Delta\varphi. \end{aligned} \tag{11.3}$$

Equation (11.3) reads that an analytic function f preserves angles at each point z_0 such that $f'(z_0) \neq 0$.

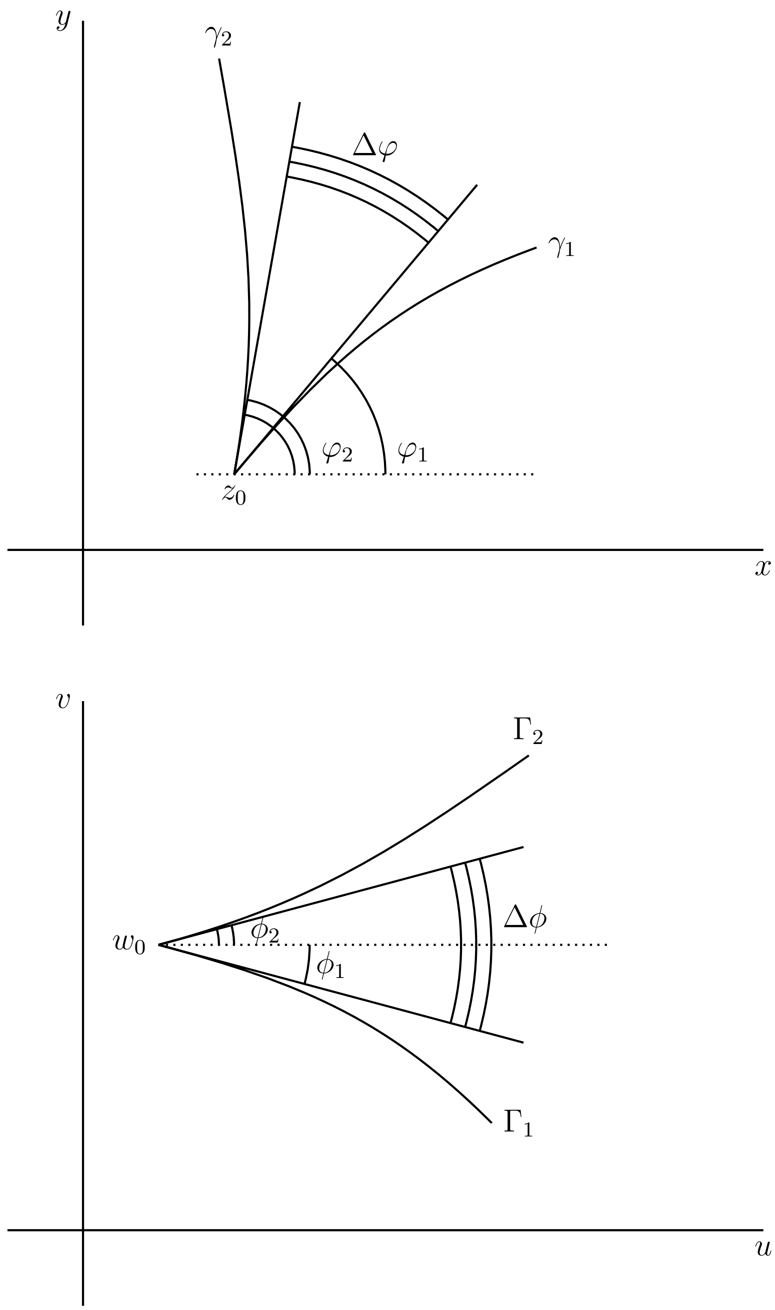


FIGURE 1. Conformal mapping: preservation of angles

Read (11.1) now differently

$$\lim_{\Delta z \to 0} \frac{\Delta w\,(z_0)}{\Delta z} = Ae^{i\alpha} \qquad \Rightarrow \qquad \Delta w = Ae^{i\alpha}\Delta z + \varepsilon(z_0, \Delta z)$$

where

$$\lim_{\Delta z \to 0} \frac{\varepsilon(z, \Delta z)}{\Delta z} = 0.$$

Recalling the definition of the differential

$$dw = Ae^{i\alpha}\Delta z \qquad \Rightarrow \qquad |dw| = A|\Delta z|.$$

Now inspect the picture below

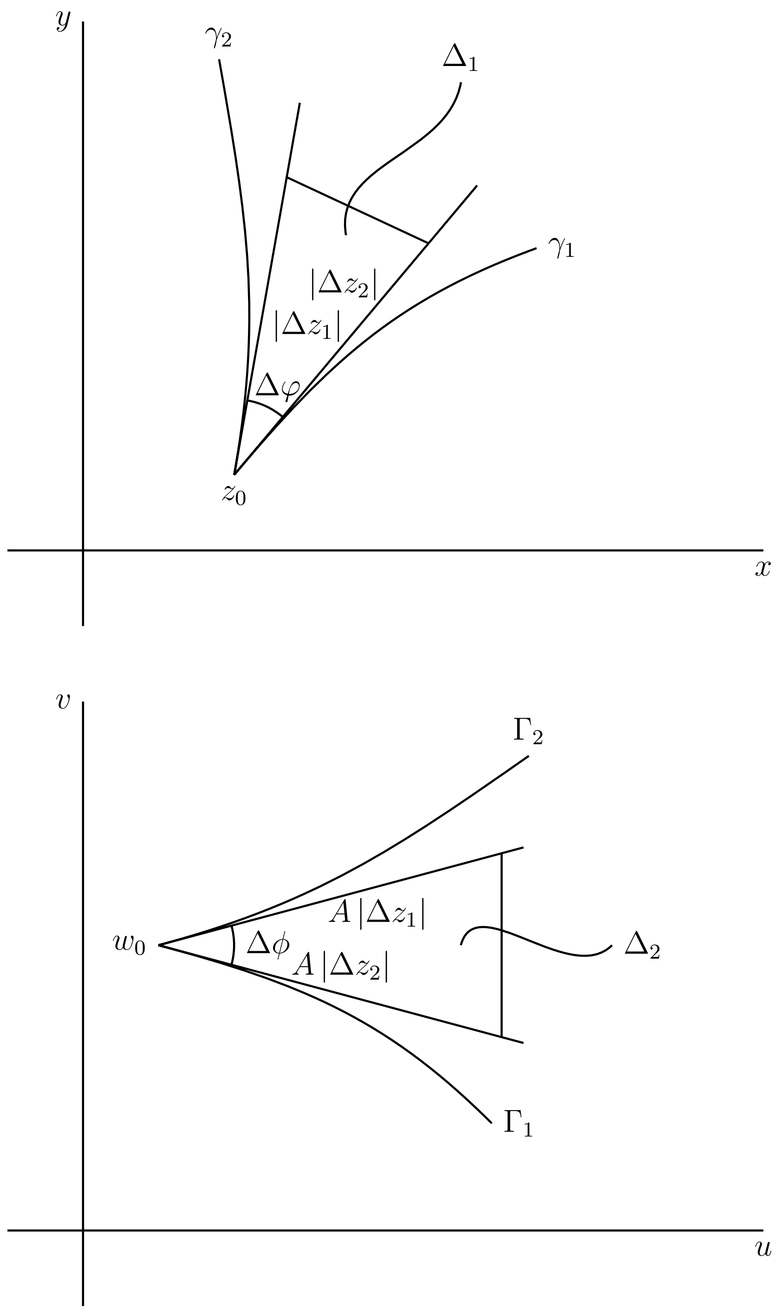


FIGURE 2. Conformal mapping: constant dilation

One can easily see from Figure 2 that triangles Δ_1 and Δ_2 are similar. In other words, for any z_0 such that $f'(z_0) \neq 0$ and a small triangle Δ_{z_0} with a vertex at z_0 the image of $f(\Delta_{z_0})$ will be a triangle "approximately" similar to Δ_{z_0}. This is a very important property of analytic functions which is called conformal. In other words, an analytic funtion preserves shapes of "small" regions.

2. Conformal Map

DEFINITION 11.1. *A map, $f : E \to f(E)$is called conformal if it preserves angles and has a constant dilation at any $z_0 \in E$.*

PROPOSITION 11.2. *An analytic function f is conformal at z_0 if and only if*

$$f'(z_0) \neq 0.$$

The proof is left as an exercise.

Hint: constant dilation at a point, z_0, by definition, means

$$\lim_{\Delta z \to 0} \left| \frac{\Delta f(z_0)}{\Delta z} \right|$$

exists.

THEOREM 11.3. *If $f : E \to \mathbb{C}$ is univalent then $f'(z_0) \neq 0$ for all $z \in E$.*

The proof is left as an exercise.

COROLLARY 11.4. *If $f : E \to \mathbb{C}$ is univalent then f is conformal.*

Exercises

Exercise 11.1 Prove Proposition 11.2. Hint: constant dilation at a point, z_0, by definition , means

$$\lim_{\Delta z \to 0} \left| \frac{\Delta f(z_0)}{\Delta z} \right|$$

exists.

Exercise 11.2 Prove Theorem 11.3*.

LECTURE 12

Möbius Transforms (Linear Fractional Transforms)

DEFINITION 12.1. *The function*

$$\varphi(z) := \frac{az+b}{cz+d}, \ \text{where } (ad \neq bc) \tag{12.1}$$

is called the <u>*Möbius transform*</u>.

PROPOSITION 12.2. *Let the Möbius transform, φ, be defined by* (12.1). *Then*

(1) *φ is analytic on $\mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$*
(2) *φ is univalent and conformal on $\mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$*
(3) *the inverse, φ^{-1} of φ, is given by*

$$\varphi^{-1}(z) = \frac{dz-b}{-cz+a}.$$

PROOF. Since (1) is clear, we will begin by considering (2). Simply differentiate (12.1) by the quotient rule to conclude that

$$\varphi'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0.$$

So, by Proposition 11.2, φ is conformal on $\mathbb{C} \setminus \left\{-\frac{d}{c}\right\}$. To prove that φ is univalent, we write

$$\begin{aligned} \frac{az+b}{cz+d} = w &\iff az+b = czw+dw \\ &\iff (a-cw)z = dw-b \\ &\iff z = \frac{dw-b}{a-cw} = \frac{dw-b}{-cw+a}, \ \forall\, w \neq \frac{a}{c}, \end{aligned} \tag{12.2}$$

which implies that φ is univalent. Thus, (2) is done. Moreover, switching $w \leftrightarrow z$ in (12.2) yields (3). □

PROPOSITION 12.3. *Let φ be a Möbius transform. Then*

$$\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1$$

where φ_1 and φ_3 are linear mappings, and φ_2 is an inversion, i.e. $\varphi_2(z) = \dfrac{1}{z}$. Moreover, if $c \neq 0$

$$\begin{aligned}\varphi_1(z) &= cz + d\\ \varphi_2(z) &= \frac{1}{z}\\ \varphi_3(z) &= \frac{a}{c} + \left(b - \frac{ad}{c}\right) z.\end{aligned}$$

If $c = 0$ then φ is linear.

The proof is left as an exercise.

The importance of the Möbius Transform is due to the following proposition.

PROPOSITION 12.4. *A Möbius transform φ transforms lines and circles to lines and circles (i.e. φ preserves lines and circles).*

PROOF. We show first that the statement holds for a linear $\varphi_1(z)$. Recall that any line L in $\mathbb{R}^2$ can be written in the parametric form $(x(t), y(t))$ where $x(t)$ and $y(t)$ are linear (real) functions of t. Consider now, $\varphi_1(z(t)) = cz(t)+d$, where $z(t) = x(t)+iy(t)$. Denoting $c = c_1 + ic_2$ and $d = d_1 + id_2$

$$\varphi_1(z(t)) = \underbrace{c_1x(t) - c_2y(t) + d_1}_{\text{linear function of } t} + i\,\underbrace{(c_2x(t) + c_1y(t) + d_2)}_{\text{linear function of } t}.$$

It follows that $\varphi_1(z(t)) := u(t) + iv(t)$ where $u(t)$ and $v(t)$ are linear functions. Hence, $\varphi_1(z(t))$ represents a line in the w-plane.

Consider now a circle $C_\rho(z_0)$ in the z-plane. It is well-known from calculus (think about it) that

$$C_\rho(z_0) = \left\{z : \ z = z_0 + \rho e^{it}, \ 0 \le t < 2\pi\right\}.$$

Thus, $z \in C_\rho(z_0) \iff z = z_0 + \rho e^{it}$ and hence

$$\underbrace{\rho_1(z_0 + \rho e^{it})}_{=:w} = c\rho e^{it} + \underbrace{d + cz_0}_{=:w_0}$$

It follows that $|w - w_0| = \rho|c|$. That is, $w \in C_{\rho|c|}(w_0)$, i.e. $\varphi_1(C_\rho(z_0)) = C_{\rho|c|}(\varphi_1(z_0))$.

It remains to show that the mapping $w = \dfrac{1}{z}$ transforms circles and lines into circles and lines. Note that when a point $w = u + iv$ is the image of a nonzero $z = x + iy$ under the transformation $w = \dfrac{1}{z}$, writing $w = \dfrac{\overline{z}}{|z|^2}$ yields that

$$u = \frac{x}{x^2 + y^2}, \ v = \frac{-y}{x^2 + y^2}. \tag{12.3}$$

Moreover, since $z = \dfrac{1}{w} = \dfrac{\overline{w}}{|w|^2}$,

$$x = \frac{u}{u^2 + v^2}, \ y = \frac{-v}{u^2 + v^2}. \tag{12.4}$$

Now note that when A, B, C, and D are all real numbers satisfying the condition $B^2 + C^2 > 4AD$, the equation

$$A(x^2 + y^2) + Bx + Cy + D = 0 \tag{12.5}$$

represents an arbitrary circle or line, where $A \neq 0$ for a circle and $A = 0$ for a line. If $A \neq 0$ it is indeed necessary for $B^2 + C^2 > 4AD$ since by completing the square we may rewrite (12.5) as

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2 + C^2 - 4AD}}{2A}\right)^2.$$

Note that if $A = 0$ then $B^2 + C^2 > 0$. Thus, either B or C is greater than 0. Now observe that if x and y satisfy (12.5), by (12.4) we may substitute. After simplification we conclude that u and v satisfy the following equation

$$D(u^2 + v^2) + Bu - Cv + A = 0, \tag{12.6}$$

which also represents a circle or line. Conversely, if u and v satisfy (12.6), it follows from (12.3) that x and y satisfy (12.5).

Now by (12.5) and (12.6) we may conclude that

(1) a circle ($A \neq 0$) not passing through the origin ($D \neq 0$) in the z-plane is transformed into a circle not passing through the origin in the w-plane;
(2) a circle ($A \neq 0$) through the origin ($D = 0$) in the z-plane is transformed into a line that does not pass through the origin in the w-plane;
(3) a line ($A = 0$) not passing through the origin ($D \neq 0$) in the z-plane is transformed into a circle through the origin in the w-plane;
(4) a line ($A = 0$) through the origin ($D = 0$) in the z-plane is transformed into a line through the origin in the w-plane.

This concludes our proof. □

It is left as an exercise to the reader to come up with a more elegant proof of Proposition 12.4.

PROPOSITION 12.5. *Given that φ_1 and φ_2 are Möbius transforms, it follows that $\varphi_2 \circ \varphi_1$ is a Möbius transform.*

The proof is left as an exercise.

PROPOSITION 12.6. *Assume $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ are sets of distinct numbers. Then there exists a Möbius transform, φ, where*

$$\varphi(z_k) = w_k, \;\; k = 1, 2, 3.$$

Moreover, φ can be explicitly constructed by

$$\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}.$$

The proof is left as an exercise.

Exercises

Exercise 12.1 Prove Proposition 12.3.

Exercise 12.2 Show that

(1) $\varphi(z) = \dfrac{i-z}{i+z}$ maps $\mathbb{C}^+$ onto $\mathbb{D}$.

(2) $\psi(z) = i\dfrac{1-z}{1+z}$ maps $\mathbb{D}$ onto $\mathbb{C}^+$.

Exercise 12.3 Prove Proposition 12.5.

Exercise 12.4 Prove Proposition 12.6.

LECTURE 13

Complex Integration

After having studied differentiation it's natural to start studying integration. Before we can start we must review some of the topics covered in Calculus.

We assume that we know what a curve in $\mathbb{R}^2$ (and thus in $\mathbb{C}$) is. We also remember what we call a piecewise smooth curve. In our course we consider only such curves. Below are some typical examples of piecewise smooth curves.

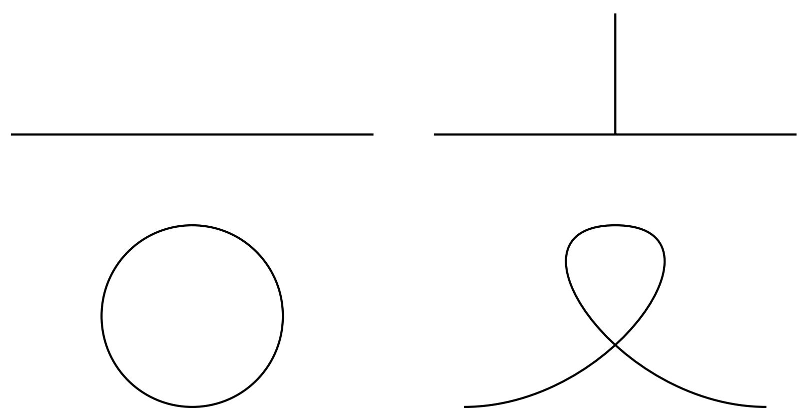


FIGURE 1. Some piecewise smooth curves

Throughout our course, given a curve C, we denote its length by $|C|$.

Recall then the concept of the path (or line) integral introduced in Calculus. There are six cases we examine to understand the path integrals.

Case 1. Let C be a piecewise smooth curve with no self-intersections (as in Figure 2) and endpoints A and B (A and B do not necessarily have to be distinct).

Let $F(x, y) = (u(x, y), v(x, y))$ be a 2-vector valued function of two real variables x and y defined on C. Assume that u and v are continuous on C. This means

$$\lim_{\substack{z\to z_0\\ z\in C}} u(z) = u(z_0).$$

where $z = (x, y) = x + iy$. Partition C as indicated in Figure 2 with $n + 1$ arbitrary, distinct points $\{z_0, z_1, \ldots, z_k, \ldots, z_n\}$ where $z_0 = A$ and $z_n = B$. Let C_k denote the kth part of C; with C_1 being the first part (i.e. the part between z_0 and z_1). Thus

$$C = \bigcup_{k=1}^{n} C_k.$$

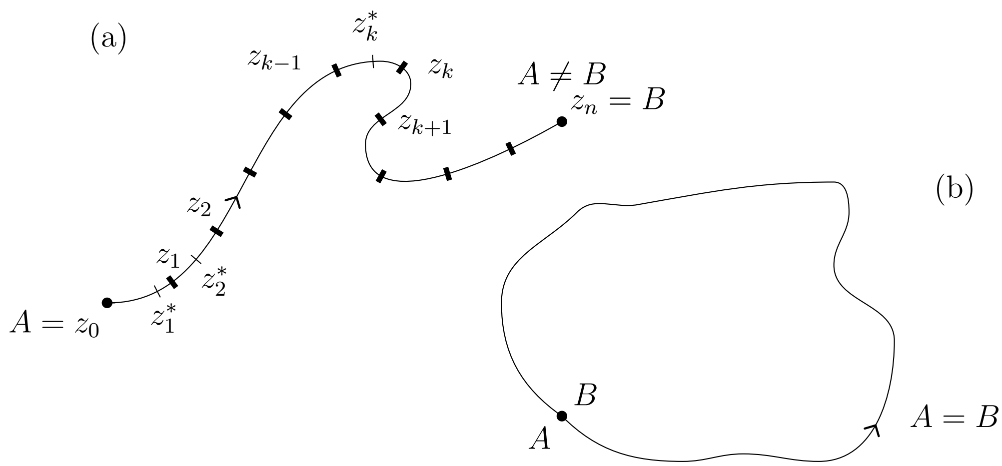


FIGURE 2. (a) A partitioned piecewise smooth curve; (b) A closed, positively oriented curve

Let $z_k^* \in C_k$ be an arbitrary point in C_k (including possibly the endpoints). Form the following Riemann sum

$$S_n := \sum_{k=1}^{n} u(z_k^*)\Delta x_k + v(z_k^*)\Delta y_k \tag{13.1}$$

where $\Delta x_k = \operatorname{Re}\Delta z_k$, $\Delta y_k = \operatorname{Im}\Delta z_k$, and $\Delta z_k = z_k - z_{k-1}$.

Denote $\rho = \max\limits_{1\le k\le n} |C_k|$. If the sum S_n converges as $\rho \to 0$ to the limit S independent of a particular choice of $\{z_k, z_k^*\}$ then we say that S is the line integral of F along C from A to B and denote this by

$$S = \int_C udx + vdy. \tag{13.2}$$

Thus

$$\int_C udx + vdy \overset{\text{def}}{=} \lim_{\rho\to 0} \sum_{k=1}^{n} u(z_k^*)\Delta x_k + v(z_k^*)\Delta y_k.$$

When we need to specify the endpoints A and B we write C_{AB}. We also agree to use the convention $C_{BA} = -C_{AB}$ or just $-C$ if we want to emphasize that we are integrating in the direction opposite of C. If we note that $udx + vdy = F \cdot dX$, the dot product of the vector F and $dX = \begin{pmatrix} dx \\ dy \end{pmatrix}$, then (13.2) reads

$$\int_C udx + vdy = \int_C FdX.$$

PROPOSITION 13.1. *Let C be a piecewise smooth curve with no self-intersections and $|C| < \infty$. If F is continuous on C then the following properties hold:*

(1) *For all constants $\alpha, \beta \in \mathbb{R}$*
$$\int_C (\alpha F_1 + \beta F_2)dX = \alpha \int_C F_1 dX + \beta \int_C F_2 dX \qquad \textit{(linearity property).}$$

(2) *Let* $C = C_1 \bigcup C_2$ *where* C_1 *and* C_2 *are disjoint. Then*
$$\int_C F dX = \int_{C_1} F dX + \int_{C_2} F dX \qquad \text{(additive property).}$$
(3) $\displaystyle\int_{-C} F dX = -\int_C F dX.$

The proof is left as an exercise.

Case 2. Let C be a piecewise smooth curve where $|C| < \infty$ but C has n self-intersections, where n is finite. In this case we break C into m curves where C_m has no self-intersections (but can be closed), $C = \bigcup_{k=1}^m C_k$, and the C_k are pairwise disjoint. The exact value of m depends on the circumstances. For example, in Figure 3 the curve on the left would be broken into three case 1 curves, C_1, C_2, and C_3 whereas the curve on the right would be broken into four case 1 curves. Set
$$\int_C F dX \stackrel{\text{def}}{=} \sum_{k=1}^m \int_{C_k} F dX.$$

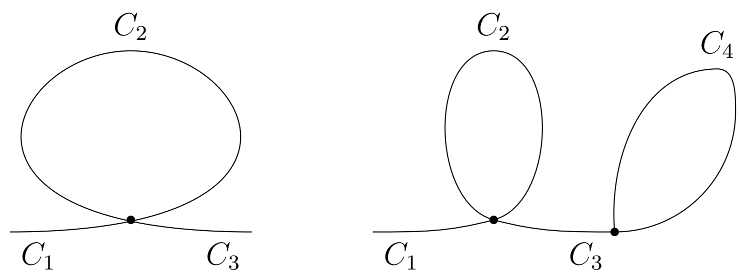


FIGURE 3. Some piecewise smooth curves with self-intersections

Case 3. Let C be a curve where $|C| < \infty$ but C is not continuous.
We then break C into continuous curves $C_1, C_2, \dots C_n$ and set
$$\int_C F dX \stackrel{\text{def}}{=} \sum_{k=1}^n \int_{C_k} F dX.$$
An example of this can be seen in Figure 4.

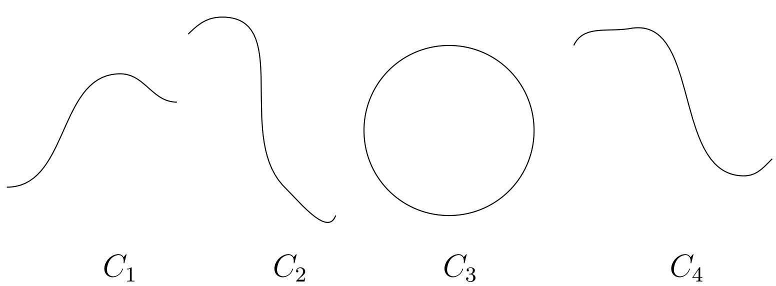


FIGURE 4. A discontinuous finite curve

Case 4. Let C be a curve where $|C| = \infty$ with one endpoint at a finite point A and the other at ∞. Denote $||B||$ as the norm of the vector B in $\mathbb{R}^2$. Then

$$\int_C FdX \overset{\text{def}}{=} \lim_{\substack{||B||\to\infty \\ B\in C}} \int_{C_{AB}} FdX$$

if this limit exists.

Case 5. Let C be a curve where $|C| = \infty$ and both endpoints are at ∞. We then select a point $B \in C$ and define

$$\int_C FdX \overset{\text{def}}{=} \lim_{\substack{||A||\to\infty \\ A\in C}} \int_{C_{AB}} FdX + \lim_{\substack{||D||\to\infty \\ D\in C}} \int_{C_{BD}} FdX$$

if both limits exist.

Case 6. Let C_n be a closed curve where C_n consists of n copies of the same closed curve C. In this case we call n the winding number of C. We can then define

$$\int_{C_n} FdX \overset{\text{def}}{=} n \int_C FdX.$$

There are some other cases but we don't care much about them in this course.

DEFINITION 13.2. *Curves which are finite combinations of curves from Cases 1-6 are called* <u>*paths*</u>.

EXAMPLE 13.3. *This is a path!*

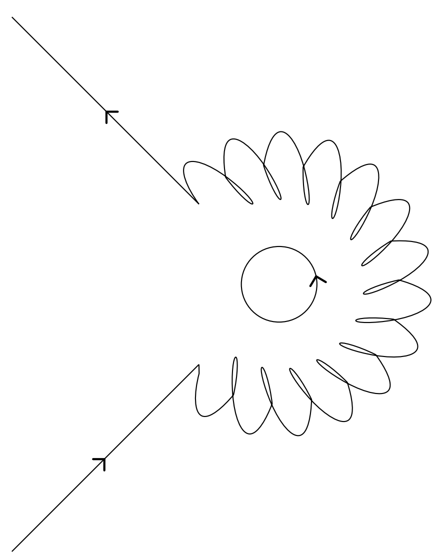


FIGURE 5. An example of a path

The following definition can be considered the main definition for this lecture.

DEFINITION 13.4. *Let C be a path and $f(z) = u(z)+iv(z)$ be a continuous function on C. Then*

$$\int_C f(z)dz \stackrel{def}{=} \int_C udx - vdy + i\int_C vdx + udy. \tag{13.3}$$

REMARK 13.5. *Equation* (13.3) *can also be written as*

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy).$$

The following theorem will play a crucial role in our course.

THEOREM 13.6 (The Triangle Inequality). *Let C be a path and $f(z)$ be a continuous function on C. Then*

$$\left|\int_C f(z)dz\right| \le \int_C |f(z)|\,|dz|\,. \tag{13.4}$$

The proof is left as an exercise.

Exercises

Exercise 13.1 Prove (2) and (3) in Proposition 13.1.

Exercise 13.2 Prove that if C is a closed curve (as in Figure 2(b)) then the integral along C doesn't depend on the initial point of integration.

Exercise 13.3 Prove Theorem 13.6.

LECTURE 14

Change of Variables; Sufficiency of the Cauchy-Riemann Conditions

1. Change of Variables

Let C be a smooth bounded path. We know from calculus that any such path can be parameterized, and the parameterizing function is continuously differentiable. In other words, if C is a smooth bounded path, then there exists a function $\gamma(t) = (\xi(t), \eta(t))$

$$\gamma : [a, b] \to C$$

such that γ is one-to-one and $\gamma' \in C[a, b]$. Note that $\gamma'(t) \neq 0$ for all $t \in (a, b)$. We also know from calculus that

$$\int_C u\ dx + v\ dy = \int_a^b [u(\gamma(t))\xi'(t) + v(\gamma(t))\eta'(t)]\ dt. \tag{14.1}$$

Observe that the integral on the right hand side of (14.1) is a real valued definite integral, something studied in Calculus I. With (14.1) in hand, by definition,

$$\begin{aligned}
\int_C f(z)\ dz &= \int_C u\ dx - v\ dy + i \int_C v\ dx + u\ dy \\
&= \int_a^b [u(\gamma(t))\xi'(t) - v(\gamma(t))\eta'(t)]\ dt + i \int_a^b [v(\gamma(t))\xi'(t) - u(\gamma(t))\eta'(t)]\ dt \\
&= \int_a^b [u(\gamma(t)) + iv(\gamma(t))][\xi'(t) + i\eta'(t)]\ dt = \int_a^b f(\gamma(t))\gamma'(t)\ dt.
\end{aligned}$$

(Note that f need not be analytic.) Thus we have

$$\int_C f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt. \tag{14.2}$$

With (14.2) in hand, we can derive the more general statement

THEOREM 14.1 (Change of variable formula). *Let f be continuous on a smooth path C, with $|C| < \infty$. Let g be analytic on a domain containing some curve Γ such that g establishes a one-to-one correspondence between the curves Γ and C. Then*

$$\int_C f(z)\ dz = \int_\Gamma f(g(\xi))g'(\xi)\ d\xi. \tag{14.3}$$

PROOF. Consider the right hand side of (14.3) and let $\gamma(t)$ be a parametrization of Γ. So

$$\int_\Gamma f(g(\xi))g'(\xi)\,d\xi = \int_a^b f(g(\gamma(t)))\quad \underbrace{g'(\gamma(t))\gamma'(t)\,dt}_{=dg(\gamma(t)),\text{ by the chain rule}}$$
$$= \int_a^b f(g(\gamma(t)))\,dg(\gamma(t)).$$

But $g \circ \gamma : [a,b] \to C$ defines a parametrization of C and, by (14.2),

$$\int_a^b f(g(\gamma(t)))\,dg(\gamma(t)) = \int_C f(z)\,dz. \qquad \square$$

REMARK 14.2. *The statement of Theorem 14.1 is not the most general but will be sufficient for our purposes.*

EXAMPLE 14.3. *The following integral is very important in complex analysis*

$$I = \int_{C_\rho(z_0)} \frac{dz}{z - z_0}.$$

Here, we integrate in the positive (counterclockwise) direction. Take the following parametrization:

$$z = z_0 + \rho e^{it}, \qquad 0 \le t < 2\pi.$$

By (14.3), *we have*

$$I = \int_0^{2\pi} \frac{\rho e^{it} i\,dt}{\rho e^{it}} = \int_0^{2\pi} i\,dt = 2\pi i.$$

Thus,

$$\int_{C_\rho(z_0)} \frac{dz}{z - z_0} = 2\pi i$$

which is independent of z_0 and ρ.

2. Sufficient Conditions of Analyticity

We proved a while ago that the Cauchy-Riemann conditions are necessary for analyticity. We now show that they are sufficient.

THEOREM 14.4. *If $u(z)$ and $v(z)$ are differentiable at $z = z_0$ (in the Calculus III sense) and*

$$u_x(z_0) = v_y(z_0)$$
$$u_y(z_0) = -v_x(z_0)$$

then $f(z) = u(z) + iv(z)$ is differentiable at $z = z_0$ (in the complex analysis sense).

PROOF. Since u, v are differentiable at $z = z_0$, then

$$\begin{aligned} \Delta u(z_0) &= u_x(z_0)\Delta x + u_y(z_0)\Delta y + \varepsilon_1(z_0, \Delta z) \\ \Delta v(z_0) &= v_x(z_0)\Delta x + v_y(z_0)\Delta y + \varepsilon_2(z_0, \Delta z) \end{aligned} \tag{14.4}$$

$$\frac{\varepsilon_k(z_0, \Delta z)}{|\Delta z|} \to 0, \ \Delta z \to 0, \text{ for } \ k = 1, 2. \tag{14.5}$$

By (14.4) we have (suppressing z_0)

$$\begin{aligned} \Delta f &= \Delta u + i\Delta v \qquad (14.6) \\ &= (u_x \Delta x + \underbrace{u_y}_{= -v_x} \Delta y) + i(v_x \Delta x + \underbrace{v_y}_{= u_x} \Delta y) + \varepsilon(\Delta z) \\ &= u_x \underbrace{(\Delta x + i\Delta y)}_{= \Delta z} + v_x \underbrace{(-\Delta y + i\Delta x)}_{= i\Delta z} + \varepsilon(\Delta z) \\ &= u_x \Delta z + i v_x \Delta z + \varepsilon(\Delta z) \\ &= (u_x + i v_x)\, \Delta z + \varepsilon(\Delta z), \end{aligned}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2$.

Dividing (14.6) by Δz, we have

$$\frac{\Delta f}{\Delta z} = u_x + i v_x + \frac{\varepsilon(\Delta z)}{\Delta z}. \tag{14.7}$$

In (14.7), take the limit as $\Delta z \to 0$. By (14.5), we get

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = u_x + i v_x$$

exists and is independent of the way $\Delta z \to 0$. By definition, f is differentiable at $z = z_0$. □

We can combine Theorem 5.12 and 14.4 in one particularly important theorem.

THEOREM 14.5. *Let $u, v \in C^1(E)$. The function $f = u + iv$ is analytic on E if and only if*

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

anywhere on E.

Exercises

Exercise 14.1 Show that the curve Γ in Theorem 14.1 must be smooth. I.e., show that if C is smooth (no angle points) then $\Gamma = g^{-1}(C)$ is also smooth.

Exercise 14.2 Show that for $n \in \mathbb{Z}$

$$\frac{1}{2\pi i} \int_{C_\rho(0)} z^n \, dz = \begin{cases} 1 & \text{for } n = -1 \\ 0 & \text{for } n \neq -1. \end{cases}$$

Exercise 14.3 Show that

$$\left| \int_{C_1(0)} e^{z^2+1} \, dz \right| \leq 2\pi e^2.$$

LECTURE 15

The Cauchy Theorem

In this lecture we finally approach the core of complex analysis.

DEFINITION 15.1. *A region $E \subseteq \mathbb{C}$ is called* <u>*simply connected*</u> *if it is connected and any closed path C in E can be* <u>*continuously deformed*</u> *into any point $z_0 \in E$.*

REMARK 15.2. *Unless otherwise stated, a region is always an open set.*

Here we choose not to give the formal definition of a <u>continuous deformation</u>. For the purposes of our course a naive understanding, which is illustrated in pictures below, is sufficient. An example of a path which can be continuously deformed is shown in Figure 1 and another path that cannot be continuously deformed is shown in Figure 2.

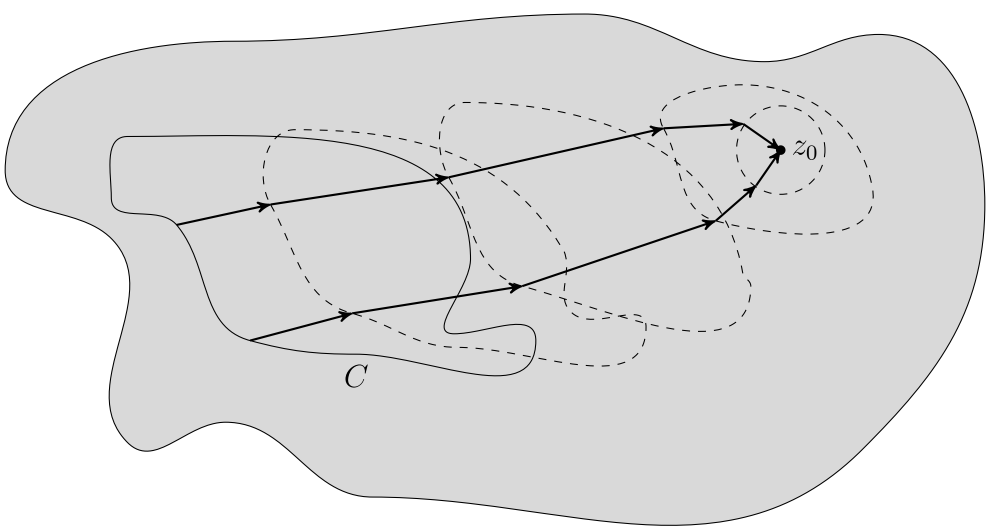


FIGURE 1. A simply connected region and continuous deformation of the closed path C.

In order to prove the Cauchy Theorem we must recall an important formula from Calculus III.

PROPOSITION 15.3 (Green's Theorem). *Let E be a simply connected region in $\mathbb{R}^2$. Suppose $E = \overline{E}$ and $P, Q \in C^1(E)$. Then*

$$\int_{\partial E} Pdx + Qdy = \iint_E (Q_x - P_y)dxdy.$$

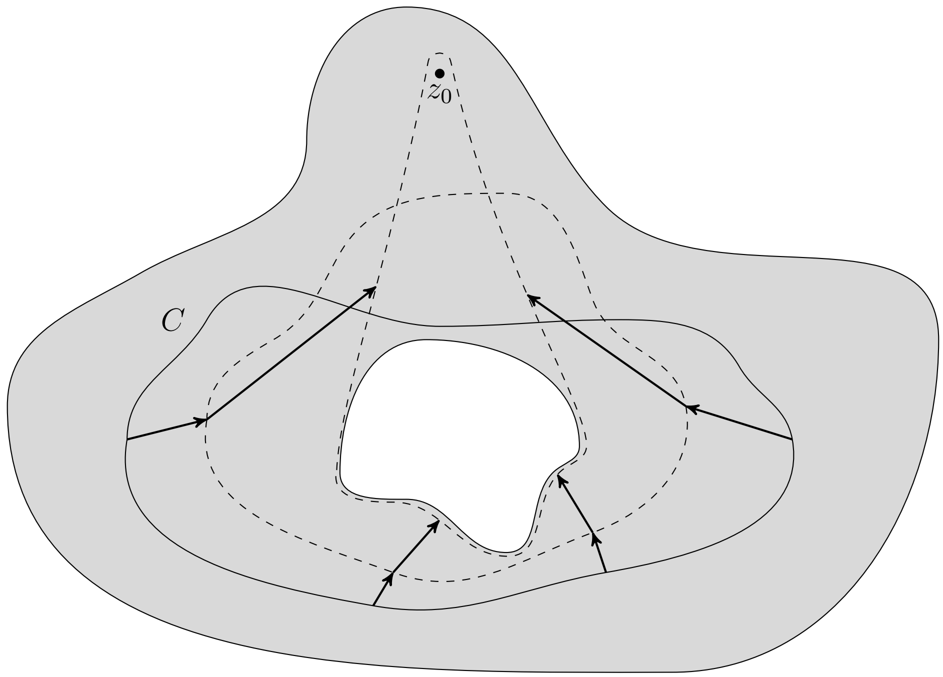


FIGURE 2. A region that is not simply connected

REMARK 15.4. *Unless otherwise stated a closed path (contour) C is a closed Case 1 curve, and we always integrate along C in the counterclockwise direction.*

THEOREM 15.5 (Cauchy's Theorem). *Let E be a simply connected region and $f(z)$ be a function that is analytic on E. Then*

$$\int_C f(z)dz = 0$$

for any closed contour $C \subset E$.

PROOF. Let C be an arbitrary closed contour in E. We denote the interior of C by $\operatorname{Int} C$. By definition,

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy.$$

By Green's Theorem we have

$$\int_C udx - vdy + i\int_C vdx + udy = \iint_{\operatorname{Int} C}(-v_x - u_y)dxdy + i\iint_{\operatorname{Int} C}(u_x - v_y)dxdy.$$

Since f is analytic on $\operatorname{Int} C \subseteq E$, $f = u + iv$ satisfies the Cauchy-Riemann conditions. That is, $u_x = v_y$ and $u_y = -v_x$. Hence

$$u_x - v_y = -u_y - v_x = 0.$$

Therefore

$$\int_C f(z)dz = \iint_{\operatorname{Int} C}(-v_x - u_y)dxdy + i\iint_{\operatorname{Int} C}(u_x - v_y)dxdy = 0. \qquad \square$$

This theorem is quite important. It has one particularly important corollary which deserves to be stated as a theorem. This theorem is also attributed to Cauchy.

THEOREM 15.6 (Cauchy). *If $f(z)$ is analytic in a simply connected region E and $f \in C(\overline{E})$ then*

$$\int_{\partial E} f(z)dz = 0. \tag{15.1}$$

This statement should be quite clear. We just apply Theorem 15.5 to $C = \partial E$. It should be noted that analyticity may fail on ∂E. It is the extra assumption of continuity on ∂E that makes Theorem 15.6 hold on ∂E. However, the proof of this relies on a specific limiting argument, which we choose to omit. With Cauchy's Theorem in hand we can quickly compute the values of contour integrals that we would only be able to estimate without the theorem.

EXAMPLE 15.7. *We consider the following integral,*

$$\int_C e^{z^2+1}dz = 0 \tag{15.2}$$

Where we integrate along the contour C which is shown in Figure 3.

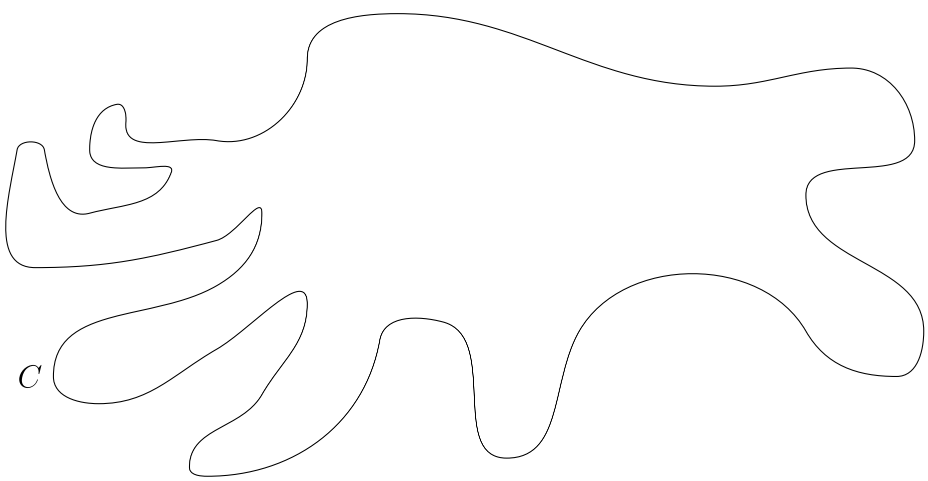


FIGURE 3. The contour C

Indeed, e^{z^2+1} is analytic on $\mathbb{C}$ and therefore analytic on $\operatorname{Int} C$*. Then* (15.2) *follows from Cauchy's Theorem (Theorem 15.5).*

COROLLARY 15.8 (Independence of Path). *Let f be analytic on a simply connected open region E. Let C_{z_0z}, C'_{z_0z} be two arbitrary continuous paths in E connecting z_0 and z. Then*

$$\int_{C_{z_0z}} f(\xi)d\xi = \int_{C'_{z_0z}} f(\xi)d\xi. \tag{15.3}$$

PROOF. Consider a closed contour C as indicated in Figure 4.

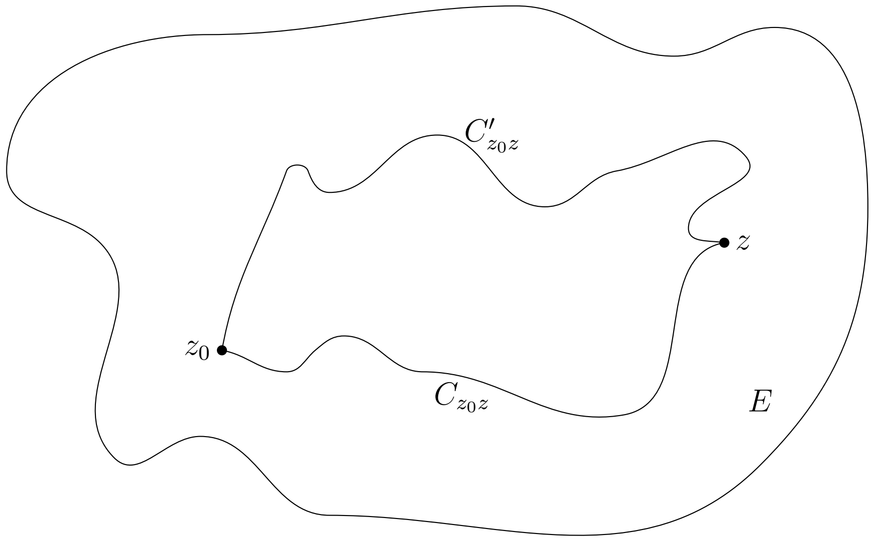


FIGURE 4. The contour C; Independence of path

Now we have $C = C_{z_0z} \cup C'_{zz_0} = C_{z_0z} \cup -C'_{z_0z}$. Since f is analytic on E, by Cauchy's Theorem, we have

$$0 = \int_C f(\xi)d\xi = \int_{C_{z_0z} \bigcup -C'_{z_0z}} f(\xi)d\xi = \int_{C_{z_0z}} f(\xi)d\xi - \int_{C'_{z_0z}} f(\xi)dz.$$

Hence

$$\int_{C_{z_0z}} f(\xi)d\xi = \int_{C'_{z_0z}} f(\xi)d\xi. \qquad \square$$

COROLLARY 15.9. *Let f be analytic on E and $z_0, z \in E$. If a path C_{z_0z} can be continuously deformed in E to another C'_{z_0z} then* (15.3) *holds.*

The proof of this corollary should be clear.

Exercises

Exercise 15.1 Let C be a positively oriented simple closed contour. Then prove that the area of $\operatorname{Int} C$ can be computed by:

$$|\text{Int}C| = \frac{1}{2i}\int_C \overline{z}dz.$$

Exercise 15.2 Show that the converse of Theorem 15.6 is false.

Exercise 15.3 Show that for any C

$$\int_C \frac{dz}{z - z_0} = \begin{cases} 2\pi i, & z_0 \in \operatorname{Int} C \\ 0, & z_0 \notin \operatorname{Int} C. \end{cases}$$

Exercise 15.4 Evaluate the integral

$$\int_C \cos z dz$$

where C is a unit semicircle in $\mathbb{C}^+$ with the beginning at $z = -1$ and end at $z = 1$.

LECTURE 16

Cauchy's Theorem for Multiply Connected Regions; Indefinite Integral

1. Cauchy's Theorem for Multiply Connected Regions

A region which is not simply connected is called multiply connected.

Two contours C_1 and C_2 are said to be (homotopically) equivalent with respect to a connected region E if one can be continuously deformed in E into the other. We then write: $C_1 \sim C_2$. A typical example can be seen in Figure 1. In the case of closed contours, they are equivalent to a point.

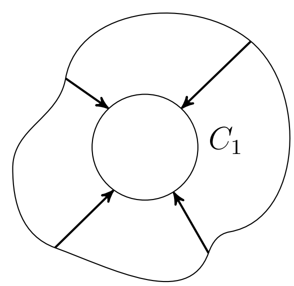


FIGURE 1. An example of equivalent contours: $C_1 \sim C_2$

A region E is called n-connected if it has n distinct holes. This of course is not a mathematically rigorous definition, but it will serve our purpose well.

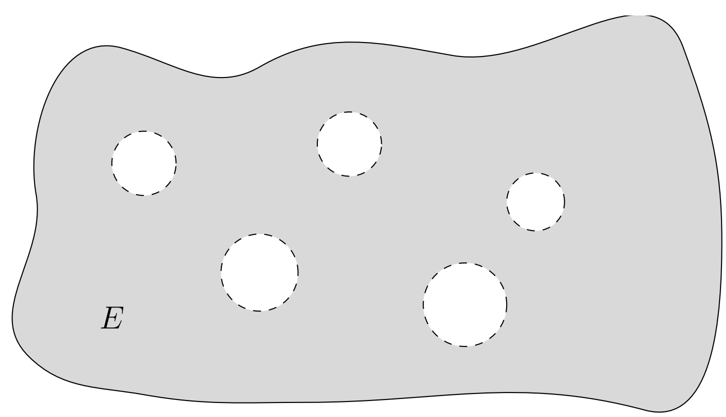


FIGURE 2. A multiply connected region – Swiss cheese

We call a closed case 1 contour a simple contour, and two simple contours C_1, C_2 are called disjoint if $\operatorname{Int} C_1 \cap \operatorname{Int} C_2 = \varnothing$.

THEOREM 16.1 (Cauchy's Theorem for Multiply Connected Regions). *Let f be analytic on a (multiply) connected domain, and let $C, C_1, C_2, \cdots, C_n$ be simply connected, positively oriented regions in E such that for all k, $\operatorname{Int} C_k \subset \operatorname{Int} C$ and all C_k are disjoint. If f is analytic on $\operatorname{Int} C \setminus \{\bigcup_{k=1}^n \operatorname{Int} C_k\}$ then*

$$\int_C f(z)dz = \sum_{k=1}^n \int_{C_k} f(z)dz. \tag{16.1}$$

PROOF. Without loss of generality we assume that each C_k encloses one hole. Consider a contour Γ as shown in Figure 3.

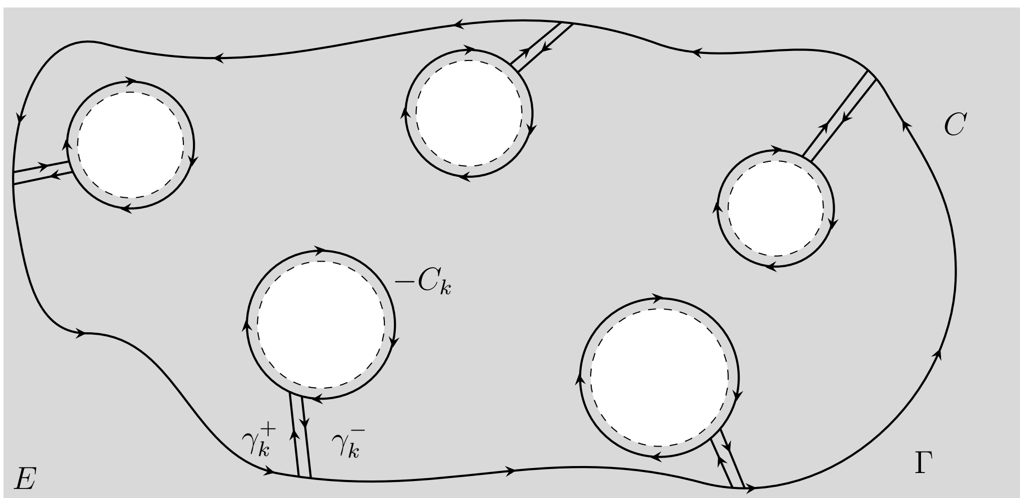


FIGURE 3. The simple closed contour Γ

$$\Gamma = C \cup \bigcup_{k=1}^n \left\{\gamma_k^+ \cup \gamma_k^- \cup (-C_k)\right\}. \tag{16.2}$$

Since $f(z)$ is analytic on $\operatorname{Int}\Gamma$, by Cauchy's Theorem, we have:

$$\int_\Gamma f(z)\, dz = 0. \tag{16.3}$$

On the other hand, by the additive property,

$$\int_\Gamma f(z)\, dz = \int_C f(z)\, dz + \underbrace{\sum_{k=1}^n \left(\int_{\gamma_k^+} f(z)\, dz + \int_{\gamma_k^-} f(z)\, dz\right)}_{=0 \text{ since } \gamma_k^- = -\gamma_k^+} + \sum_{k=1}^n \int_{-C_k} f(z)\, dz$$

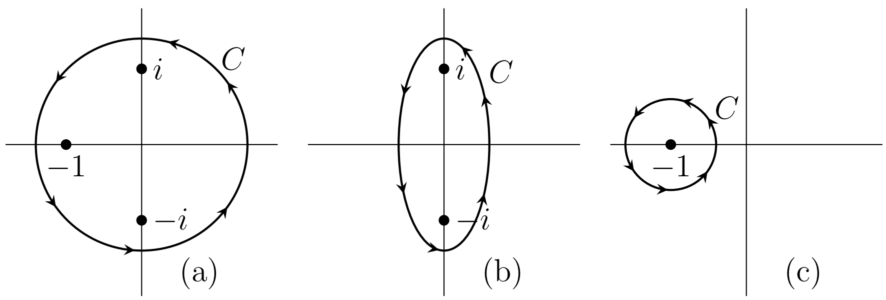
$$= \int_C f(z)\, dz - \sum_{k=1}^n \int_{C_k} f(z)\, dz. \tag{16.4}$$

Comparing (16.3) and (16.4) yields (16.1). $\square$

There is another version of Theorem 16.1 along the same lines as Theorem 15.6 presented here as a remark.

Remark 16.2. *If we assume in Theorem 16.1 that $f \in C(\overline{E})$, then the contours C_k can be deformed to the very boundary ∂E.*

Example 16.3. *Evaluate* $\displaystyle\int_C \frac{dz}{(z+1)(z^2+1)}$ *for the following contours.*



Solution We note that $f(z) = \dfrac{1}{(z+1)(z^2+1)}$ is analytic on $\mathbb{C} \setminus \{-1, i, -i\}$, and:

$$f(z) = \frac{1}{(z+1)(z^2+1)} = \frac{\frac{1}{2}}{z+1} + \frac{\frac{1}{(1+i)2i}}{z-i} + \frac{\frac{1}{(1-i)(-2i)}}{z+i}$$
$$= \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{i-1}\frac{1}{z-i} - \frac{1}{1+i}\frac{1}{z+i}\right).$$

We construct positively oriented circles of radius ρ, and respectively of centers $z_1 = -1, z_2 = i$, and $z_3 = -i$. For a small enough ρ (in our example $\rho = \frac{1}{4}$ would be sufficiently small), all C_ρ are disjoint and therefore for $k = 1, 2, 3$, and a_k being the coefficients in the partial function decomposition, we have:

$$\int_{C_\rho(z_k)} f(z)dz = \int_{C_\rho(z_k)} \frac{a_1 dz}{z-z_1} + \int_{C_\rho(z_k)} \frac{a_2 dz}{z-z_2} + \int_{C_\rho(z_k)} \frac{a_3 dz}{z-z_3} \quad \text{by linearity}$$
$$= \int_{C_\rho(z_k)} \frac{a_k dz}{z-z_k} \tag{16.5}$$

by Cauchy's Theorem since we have $z_j \notin \operatorname{Int} C_\rho(z_k)$ whenever $j \neq k$, and thus $\displaystyle\int_{C_\rho(z_k)} \frac{a_j dz}{z-z_j} = 0$.

Furthermore, from Example 14.3, we have:

$$\int_{C_\rho(z_k)} \frac{a_k dz}{z-z_k} = 2\pi i a_k. \tag{16.6}$$

(a) All $C_\rho(z_k)$ defined above are in $\operatorname{Int} C$. Hence, by applying Theorem 16.1, we obtain that:

$$\begin{aligned}\int_C \frac{dz}{(z+1)(z^2+1)} &\overset{(16.5)}{=} \frac{1}{2}\left(\int_{C_\rho(-1)} \frac{dz}{z+1} + \frac{1}{i-1}\int_{C_\rho(i)} \frac{dz}{z-i} - \frac{1}{1+i}\int_{C_\rho(-i)} \frac{dz}{z+i}\right)\\ &\overset{(16.6)}{=} \frac{1}{2}\left(2\pi i + \frac{1}{i-1}2\pi i - \frac{1}{1+i}2\pi i\right)\\ &= \pi i\left(1+\frac{1}{i-1}-\frac{1}{1+i}\right) = \pi i\left(1+\frac{1+i-i+1}{(i-1)(1+i)}\right)\\ &= \pi i(1-1) = 0.\end{aligned}$$

(b) In this case, only $C_\rho(i)$ and $C_\rho(-i)$ are in $\operatorname{Int} C$, hence:

$$\begin{aligned}\int_C \frac{dz}{(z+1)(z^2+1)} &= \frac{1}{2}\left(\frac{1}{i-1}\int_{C_\rho(i)} \frac{dz}{z-i} - \frac{1}{1+i}\int_{C_\rho(-i)} \frac{dz}{z+i}\right)\\ &= \frac{1}{2}2\pi i\left(\frac{1}{i-1}-\frac{1}{1+i}\right) = -\pi i.\end{aligned}$$

(c) Now, only $C_\rho(-1)$ is in $\operatorname{Int} C$. Therefore,

$$\begin{aligned}\int_C \frac{dz}{(z+1)(z^2+1)} &= \frac{1}{2}\int_{C_\rho(-1)} \frac{dz}{z+1}\\ &= \frac{1}{2}2\pi i = \pi i.\end{aligned}$$

2. Indefinite Integral

If f is analytic in a simply connected region E and $z_0, z \in E$, then

$$\int_{C_{z_0 z}} f(\xi)d\xi = \int_{C'_{z_0 z}} f(\xi)d\xi \quad \text{if} \quad C_{z_0 z} \sim C'_{z_0 z}.$$

I.e. $\displaystyle\int_{C_{z_0 z}} f(\xi)d\xi$ is determined by z_0, z and not $C_{z_0 z}$. This allows us to introduce

$$F(z) = \int_{z_0}^{z} f(\xi)d\xi$$

which we call the indefinite integral of f (or antiderivative).

THEOREM 16.4. *Let E be a simply connected region. Assume $f \in C(E)$ and $\displaystyle\int_C f(\xi)d\xi = 0$ for any simple contour C in E. Then*

$$F(z) = \int_{z_0}^{z} f(\xi)d\xi$$

is analytic on E, and

$$F'(z) = f(z) \quad \forall z \in E.$$

PROOF. Consider any $z, z+\Delta z \in E$. We define:

$$\begin{aligned}\Delta F(z) = F(z+\Delta z) - F(z) &= \int_{z_0}^{z+\Delta z} f(\xi)d\xi - \int_{z_0}^{z} f(\xi)d\xi \\ &= \int_{z_0}^{z} f(\xi)d\xi + \int_{z}^{z+\Delta z} f(\xi)d\xi - \int_{z_0}^{z} f(\xi)d\xi = \int_{z}^{z+\Delta z} f(\xi)d\xi\end{aligned}$$

where the integration is chosen along the segment $[z, z+\Delta z]$. Then,

$$\begin{aligned}\Delta F(z) - f(z)\Delta z &= \int_{z}^{z+\Delta z} f(\xi)d\xi - f(z)\int_{z}^{z+\Delta z} d\xi \\ &= \int_{z}^{z+\Delta z} (f(\xi) - f(z))d\xi \\ \overset{\text{triangle ineq}}{\Longrightarrow} |\Delta F(z) - f(z)\Delta z| &\le \int_{z}^{z+\Delta z} |f(\xi) - f(z)|\ |d\xi| \\ &\le \max_{\xi\in[z,z+\Delta z]} |f(\xi) - f(z)|\ |\Delta z| \\ \Rightarrow \left|\frac{\Delta F(z)}{\Delta z} - f(z)\right| &\le \max_{\xi\in[z,z+\Delta z]} |f(\xi) - f(z)| \underset{|\Delta z|\to 0}{\longrightarrow} 0\end{aligned}$$

since f is continuous.

Hence, by definition, $F(z)$ is differentiable, with $F'(z) = f(z)$, and since $f(z)$ is continuous, $F(z)$ is analytic on E. □

Exercises

Exercise 16.1 Let C be a positively oriented simple closed contour, and $\alpha, \beta \notin C$. What are the possible values of

$$\int_C \frac{dz}{(z-\alpha)(z-\beta)}\ ?$$

Exercise 16.2 Show that if $p(z)$ is a polynomial and $z_0 \in \operatorname{Int} C$, where C is a positively oriented simple closed contour, then

$$\frac{1}{2\pi i}\int_C \frac{p(z)dz}{z-z_0} = p(z_0).$$

Note that this is a particular case of the Cauchy formula to be studied in the next lecture.

LECTURE 17

The Log Function Revisited; The Cauchy Formula

1. The Log Function Revisited

Theorem 16.4 offers a new definition of the log function.

DEFINITION 17.1. *The function* $f(z)$ *defined by*

$$f(z) = \int_1^z \frac{d\xi}{\xi} \tag{17.1}$$

is called the logarithm of z, $\log z$.

By Theorem 16.4, $f(z)$ is analytic on any simply connected region excluding 0. If $z = x, x > 0$, then

$$f(x) = \int_1^x \frac{d\xi}{\xi} = \ln x.$$

2. The Cauchy Formula

This formula is the central formula of Complex Analysis.

THEOREM 17.2 (The Cauchy Formula). *Let* f *be analytic on* E *and* $C \subset E$ *be a positively oriented simple contour. Then*

$$\forall\, z_0 \in \operatorname{Int} C \qquad \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = f(z_0). \tag{17.2}$$

PROOF. Since $\dfrac{f(z)}{z - z_0}$ is analytic on $E \backslash \{z_0\}$, $C \sim C_\rho(z_0)$ for some ρ (see Figure 1).

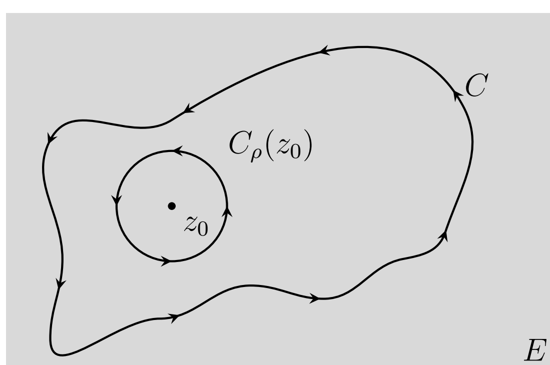


FIGURE 1. The equivalent contours C and $C_\rho(z_0)$

By Cauchy's Theorem

$$\int_C \frac{f(z)dz}{z-z_0} = \int_{C_{\rho(z_0)}} \frac{f(z)dz}{z-z_0} = \underbrace{\int_{C_{\rho(z_0)}} \frac{f(z)-f(z_0)dz}{z-z_0}}_{=I_\rho} + f(z_0) \underbrace{\int_{C_{\rho(z_0)}} \frac{dz}{z-z_0}}_{=2\pi i}. \tag{17.3}$$

For I_ρ in (17.3) we have

$$\begin{aligned} |I_\rho| &\overset{\text{triangle ineq}}{\leq} \int_{C_{\rho(z_0)}} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| |dz| \\ &\leq \max_{z \in C_{\rho(z_0)}} |f(z)-f(z_0)| \cdot \underbrace{\int_{C_{\rho(z_0)}} \frac{|dz|}{|z-z_0|}}_{=2\pi} \quad \to 0 \quad \text{as} \quad \rho \to 0. \end{aligned} \tag{17.4}$$

Taking $\rho \to 0$ in (17.3), we have

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0) + \underbrace{\lim_{\rho \to 0} \int_{C_{\rho(z_0)}} \frac{f(z)-f(z_0)}{z-z_0} dz}_{=0 \text{ by } (17.4)}.$$

□

REMARK 17.3. *If in Theorem 17.2 the region E is bounded and the function f is continuous on ∂E then the contour C can be pushed all the way to ∂E. This simple observation will be very important in the future.*

COROLLARY 17.4 (Gauss Mean Value Theorem). *If f is analytic on $\mathbb{D}_\rho(z_0)$ and continuous on $\overline{\mathbb{D}}_\rho(z_0)$ then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0 + \rho e^{i\theta}\right) d\theta. \tag{17.5}$$

PROOF. Taking in (17.2) $C = C_\rho(z_0)$ we have

$$f(z_0) = \frac{1}{2\pi i} \int_{C_{\rho(z_0)}} \frac{f(z)}{z-z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} f\left(z_0+\rho e^{i\theta}\right) d\theta.$$

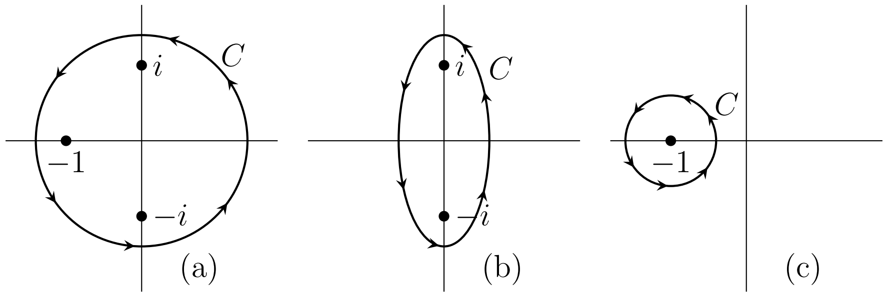
□

Formula (17.5) says that the value of an analytic function at the center of a disk is determined by its values on the circumference.

EXAMPLE 17.5. *As in Example 16.3, we will evaluate $\int_C \frac{dz}{(z+1)(z^2+1)}$ for various contours, but this time using the Cauchy formula.*

Solution Let

$$f_1(z) = \frac{1}{z^2+1}, \qquad f_2(z) = \frac{1}{(z+1)(z+i)}, \text{ and} \qquad f_3(z) = \frac{1}{(z+1)(z-i)}.$$



As before, we construct the following positively oriented circles: $C_1 = C_\rho(-1)$, $C_2 = C_\rho(i)$, and $C_3 = C_\rho(-i)$ where ρ is small enough so that C_1, C_2, and C_3 are disjoint. We also note that thus f_k is analytic on $C_k \cup \operatorname{Int} C_k$ for $k = 1, 2, 3$.

(a) All C_k defined above are in $\operatorname{Int} C$. Hence, by applying Theorem 16.1, we obtain that:

$$\begin{aligned}\int_C \frac{dz}{(z+1)(z^2+1)} &= \int_{C_1} \frac{f_1(z)}{z+1} dz + \int_{C_2} \frac{f_2(z)}{z-i} dz + \int_{C_3} \frac{f_3(z)}{z+i} dz \\ &\overset{\text{Cauchy's formula}}{=} 2\pi i \left(f_1(-1) + f_2(i) + f_3(-i)\right) \\ &= 2\pi i \left(\frac{1}{2} + \frac{1}{(i-1)(2i)} + \frac{1}{(-1+i)(-2i)}\right) \\ &= \pi i \left(1 + \frac{1}{i-1} - \frac{1}{1+i}\right) = \pi i (1-1) = 0.\end{aligned}$$

(b) In this case, only C_2 and C_3 are in $\operatorname{Int} C$, hence:

$$\begin{aligned}\int_C \frac{dz}{(z+1)(z^2+1)} &= \int_{C_2} \frac{f_2(z)}{z-i} dz + \int_{C_3} \frac{f_3(z)}{z+i} dz \\ &\overset{\text{Cauchy's formula}}{=} 2\pi i \left(f_2(i) + f_3(-i)\right) \\ &= \frac{1}{2} 2\pi i \left(\frac{1}{i-1} - \frac{1}{1+i}\right) = -\pi i.\end{aligned}$$

(c) Now, since $C_1 \sim C$, we have:

$$\int_C \frac{dz}{(z+1)(z^2+1)} = \int_{C_1} \frac{f_1(z)}{z+1} dz \overset{\text{Cauchy's formula}}{=} 2\pi i f_1(-1) = 2\pi i \frac{1}{2} = \pi i.$$

Exercises

Exercise 17.1 Show that Definition 17.1 is in agreement with Definition 9.5 with $\arg z \in (-\pi, \pi]$ and the cut along $\mathbb{R}_-$.

Exercise 17.2 Let f, g be analytic on E and $f(z) = g(z)\ \forall\ z \in C$ where C is a simple closed contour. Show that $f(z) = g(z)\ \forall z \in \operatorname{Int} C$.

Exercise 17.3 Let f be analytic on E and $C_1(0) \subset E$. Show that

$$\frac{1}{2\pi i}\int_{C_1(0)} \frac{f(\xi)}{(\xi - z)\,\xi} d\xi = \frac{f(z) - f(0)}{z} \quad \forall z \in \mathbb{D}, z \neq 0.$$

LECTURE 18

The Maximum Modulus Principle

In this lecture we start collecting some of the crop that Cauchy's formula produces.

THEOREM 18.1 (Maximum Modulus Principle). *Let f be an analytic function on a bounded connected region E and $f \in C(\overline{E})$, then $|f(z)|$ attains its maximum on the boundary ∂E, or else $|f(z)| = \text{constant}$ on E.*

PROOF. Assume to the contrary, that a maximum value M of $|f(z)|$ is attained at some $z_0 \in E$ but $|f(z)| \neq$ constant on E, then

$$M = |f(z_0)| \geq |f(z)| \quad \forall z \in E.$$

Since $z_0 \in E$ (as opposed to ∂E), for some $\rho > 0$, $\mathbb{D}_\rho(z_0) \subset E$. By Cauchy's formula,

$$\left| \int_{C_\rho(z_0)} \frac{f(z)}{z - z_0} dz \right| = 2\pi |f(z_0)| = 2\pi M. \tag{18.1}$$

On the other hand,

$$\left| \int_{C_\rho(z_0)} \frac{f(z)}{z - z_0} dz \right| \overset{\text{triangle ineq}}{\leq} \int_0^{2\pi} \underbrace{\left| f(z_0 + \rho e^{i\theta}) \right|}_{\leq M \text{ since } z_0 \text{ is a max}} d\theta \leq 2\pi M. \tag{18.2}$$

Combining (18.1) and (18.2) yields

$$2\pi M \leq \int_0^{2\pi} \left| f(z_0 + \rho e^{i\theta}) \right| d\theta \leq 2\pi M.$$

Hence

$$\int_0^{2\pi} \left| f(z_0 + \rho e^{i\theta}) \right| d\theta = 2\pi M. \tag{18.3}$$

We now show that (18.3) implies $|f(z)| = M$, $\forall z \in C_\rho(z_0)$. Assume to the contrary, that

$$\exists\ \xi_0 = z_0 + \rho e^{i\theta_0} \in C_\rho(z_0) : \quad |f(\xi_0)| < M.$$

Since f is continuous, there exists $[\theta_1, \theta_2]$ (with $\theta_0 \in [\theta_1, \theta_2]$), and there exists $\varepsilon > 0$ such that

$$|f(z_0 + \rho e^{i\theta})| \leq M - \varepsilon \quad \forall\, \theta \in [\theta_1, \theta_2].$$

By additivity, for the integral on the LHS of (18.3) we have

$$\begin{aligned}\int_0^{2\pi} |f| &= \int_{[\theta_1,\theta_2]} |f| + \int_{[0,2\pi)\setminus[\theta_1,\theta_2]} |f| \\ &\le (M-\varepsilon)(\theta_2-\theta_1) + M(2\pi - (\theta_2-\theta_1)) \\ &= 2\pi M - \varepsilon(\theta_2-\theta_1) < 2\pi M,\end{aligned}$$

which contradicts (18.3).

Thus, we have proved that

$$|f(z)| = M, \quad \forall z \in C_\rho(z_0), \quad \forall\ 0 < \rho < \operatorname{dist}(z_0, \partial E). \tag{18.4}$$

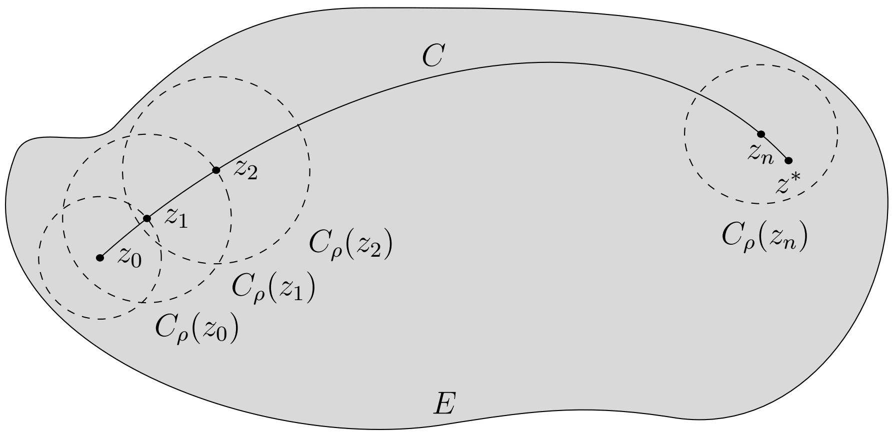


FIGURE 1. A path C from z_0 to z^*

We now show that $|f(z^*)| = M$, $\forall\ z^* \in E$. To this end, consider the path C on Figure 1. By (18.4), $|f(z_1)| = M$. By repeating the above argument for z_1 instead of z_0, we immediately conclude that $|f(z)| = M$, $\forall\ z \in C_{\rho_1}(z_1)$, $\forall\ 0 < \rho_1 < \operatorname{dist}(z_1, \partial E)$. We now continue in the same fashion. It is clear that, if at each step we choose $\rho_k = \frac{1}{2}\operatorname{dist}(z_k, \partial E)$, then it will take a finite number of steps to reach the neighborhood of z_n where z^* is. Thus $|f(z^*)| = M$. So we get a contradiction.

□

COROLLARY 18.2 (Minimum Modulus Principle). *Let f be an analytic function on a bounded connected region E and $f \in C(\overline{E})$. If $f(z) \neq 0\ \forall\ z \in E$ then $|f(z)|$ attains its minimum on the boundary ∂E, otherwise $|f(z)| = \text{constant}$ on E.*

The proof is left as an exercise.

REMARK 18.3. *In the Maximum/Mininum Modulus Principle the last part "or else $|f(z)| = \text{constant}$ on E" can be replaced with "or else $f(z) = \text{constant}$ on E".*

The proof is left as an exercise.

REMARK 18.4. *If in the Maximum/Minimum Modulus Principle f is not analytic then the statement fails. A counterexample is produced by $f(z) = 1/z$ on $\mathbb{D}$. By the same reason, the Minimum Modulus Principle fails if f has a zero in E. Take $f(z) = z$.*

REMARK 18.5. *If E is unbounded then the Maximum/Minimum Modulus Principle no longer holds. Indeed, take $f(z) = \sin z$ and $E = \mathbb{C}^+$. The function* $\sin z$ *is analytic on E and continuous on $\overline{E} = \mathbb{C}^+ \cup \mathbb{R}$. On ∂E (that is $\mathbb{R}$), $|\sin z| \le 1$. However, if $z = iy$ then*

$$|\sin z| = \left|\frac{e^{i(iy)} - e^{-i(iy)}}{2i}\right| = \left|\frac{e^{-y} - e^{y}}{2i}\right| = \frac{e^{y} - e^{-y}}{2} \to \infty, \text{ as } y \to \infty.$$

So in this case, $|f(z)|$ does not attain a maximum on either E or ∂E.

REMARK 18.6. *Connectedness is also essential in the Maximum/Minimum Modulus Principle. Indeed take $E = E_1 \cup E_2$, where $E_1 = \mathbb{D}$, $E_2 = \mathbb{D}_4 \setminus \overline{\mathbb{D}}_3$ as on Figure 2.*

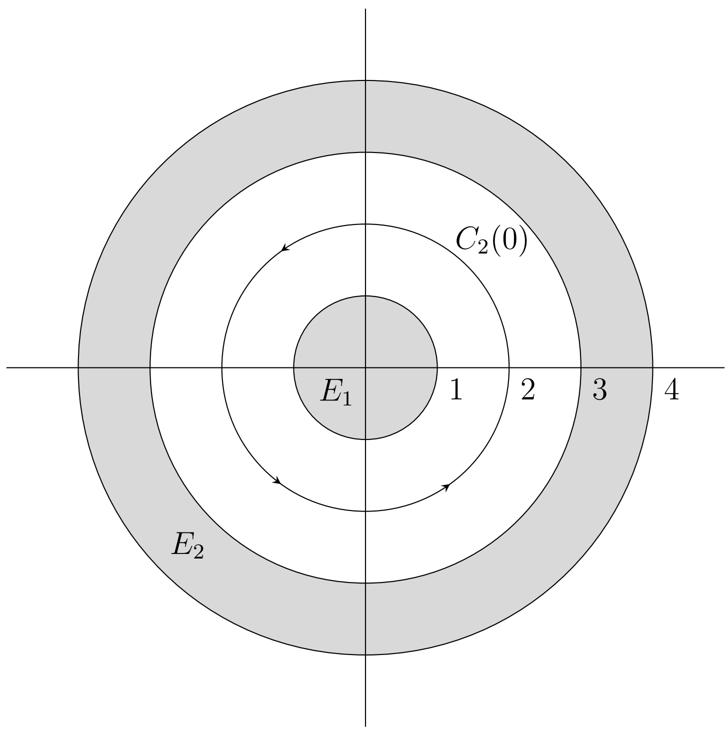


FIGURE 2. An unconnected region $E = E_1 \cup E_2$

Now consider

$$f(z) = \frac{1}{2\pi i} \int_{C_2(0)} \frac{d\xi}{\xi - z}.$$

The region E is bounded but not connected. The function f is analytic on E and continuous on $\overline{E}$. Moreover, as we know

$$f(z) = \begin{cases} 1, & z \in E_1 \\ 0, & z \in E_2. \end{cases}$$

Notice that $\max |f(z)| = 1$ *and it is attained at any point $z \in E_1$, but* $|f(z)| \neq$ constant *on E.*

REMARK 18.7. *In the Maximum/Minimum Modulus Principle, $|f(z)|$ cannot attain even a local maximum in E. Indeed, if $z_0 \in E$ were a local maximum of $|f(z)|$ then by the Maximum Modulus Principle $|f(z)|$ would be constant on some $C_\rho(z_0)$. Then this property would spread through the whole of E.*

The following versions of the Maximum/Minimum Modulus Principle are useful.

COROLLARY 18.8. *Suppose E is a bounded connected region and f is analytic on E and $f \in C(\overline{E})$. Then*

(1) *$|f|$ attains its maximum M on ∂E,*
(2) *either f is constant or $|f(z)| < M \; \forall z \in E$.*

The proof is left as an exercise.

COROLLARY 18.9. *Suppose E is a bounded connected region and f is analytic on E, $f \in C(\overline{E})$, and $f(z) \neq 0 \; \forall z \in E$. Then*

(1) *$|f|$ attains its minimum m and maximum M on ∂E,*
(2) *either f is constant or $m < |f(z)| < M \; \forall z \in E$.*

The proof is left as an exercise.

Exercises

Exercise 18.1 Prove Corollary 18.2, the Minimum Modulus Principle.

Exercise 18.2 Let f be analytic on E, prove that if $|f(z)| = \text{constant}$ on E then $f(z) = \text{constant}$.

Exercise 18.3 Prove Corollary 18.8.

Exercise 18.4 Prove Corollary 18.9.

Exercise 18.5 Let f be analytic on a region E then $\forall z_0 \in E$, the function g_{z_0} defined as

$$g_{z_0}(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0}, & z \neq z_0 \\ f'(z_0), & z = z_0 \end{cases}$$

is analytic on E.

Exercise 18.6 (Schwarz's Lemma) Prove that if f is analytic on $\mathbb{D}$, $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, and $f(0) = 0$ then $|f'(0)| \leq 1$ and $|f(z)| \leq |z| \quad \forall z \in \mathbb{D}$ (hint: use the previous exercise).

LECTURE 19

An Analytic Function is Infinitely Differentiable

In this lecture we get yet another portion of consequences of the Cauchy formula.

LEMMA 19.1 (Binomial Series). *For all $z \in \mathbb{D}$ we have*

$$\frac{1}{(1-z)^n} = \sum_{k\geq 0} \frac{(n+k-1)!}{k!(n-1)!} z^k. \tag{19.1}$$

PROOF. Consider the geometric series

$$\frac{1}{1-z} = \sum_{k\geq 0} z^k$$

which as we know is uniformly convergent on $\mathbb{D}_\rho(0)$ for $\rho < 1$. By Corollary 6.5, $\dfrac{1}{1-z}$ is infinitely differentiable and

$$\begin{aligned}\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{1-z}\right) &= \sum_{k=\geq n-1} k(k-1)\dots(k-n+2)z^{k-n+1} \\ &\overset{\text{reindexing}}{=} \sum_{k\geq 0}(k+1)(k+2)\dots(k+n-1)z^k \\ &= \sum_{k\geq 0}\frac{(k+n-1)!}{k!}z^k.\end{aligned}$$

But

$$\frac{d^{n-1}}{dz^{n-1}}\left(\frac{1}{1-z}\right) = \frac{(n-1)!}{(1-z)^n}$$

and (19.1) follows. □

THEOREM 19.2. *Let f be analytic on a simply connected region E and $C \subset E$ be a positively oriented simple contour. Then f is differentiable on E infinitely many times and*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi. \tag{19.2}$$

PROOF. We will proceed by induction. For $n = 0$ (19.2) becomes the Cauchy formula. Assume then that (19.2) holds for $n-1$. We need to prove that (19.2) holds for n. To this end consider

$$\frac{f^{(n-1)}(z+\Delta z) - f^{(n-1)}(z)}{\Delta z} - \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

By the induction hypothesis we have

$$f^{(n-1)}(z+\Delta z) - f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_C \left[\frac{1}{(\xi - z - \Delta z)^n} - \frac{1}{(\xi - z)^n}\right] f(\xi)d\xi$$

and hence

$$\begin{aligned}
&\frac{f^{(n-1)}(z+\Delta z) - f^{(n-1)}(z)}{\Delta z} - \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}}d\xi \\
&= \frac{(n-1)!}{2\pi i \Delta z}\int_C \underbrace{\left[\frac{1}{(\xi - z - \Delta z)^n} - \frac{1}{(\xi - z)^n} - \frac{n\Delta z}{(\xi - z)^{n+1}}\right]}_{=:\varphi(\alpha,\Delta z),\alpha:=\xi - z} f(\xi)d\xi.
\end{aligned} \tag{19.3}$$

For φ we have

$$\begin{aligned}
\varphi(\alpha,\Delta z) &= \frac{1}{\alpha^n}\left[\frac{\alpha^n}{(\alpha-\Delta z)^n} - 1 - \frac{n\Delta z}{\alpha}\right] \\
&= \frac{1}{\alpha^n}\left[\frac{1}{(1-\frac{\Delta z}{\alpha})^n} - 1 - \frac{n\Delta z}{\alpha}\right] \\
&= \frac{1}{\alpha^n}\left[\sum_{k\geq 0}\frac{(k+n-1)!}{k!(n-1)!}\frac{\Delta z^k}{\alpha^k} - 1 - \frac{n\Delta z}{\alpha}\right] \\
&= \frac{1}{\alpha^n}\sum_{k\geq 2}\frac{(k+n-1)!}{k!(n-1)!}\frac{\Delta z^k}{\alpha^k} \overset{\text{reindexing}}{=} \frac{\Delta z^2}{\alpha^{n+2}}\sum_{k\geq 0}\frac{(k+n+1)!}{(k+2)!(n-1)!}\frac{\Delta z^k}{\alpha^k}.
\end{aligned}$$

Hence

$$\begin{aligned}
|\varphi| &= \left|\frac{\Delta z^2}{\alpha^{n+2}}\sum_{k\geq 0}\frac{(k+n+1)!}{(k+2)!(n-1)!}\frac{\Delta z^k}{\alpha^k}\right| \\
&\leq \frac{|\Delta z|^2}{|\alpha|^{n+2}}\sum_{k\geq 0}\frac{(k+n+1)!}{(k+2)!(n-1)!}\frac{|\Delta z^k|}{|\alpha^k|}.
\end{aligned}$$

Now, let $d = \operatorname{dist}(z, C)$ and choose $|\Delta z| < \frac{1}{3}d$. Then for all $\xi \in C$ we have $|\xi - z| \geq d$. This gives us

$$\frac{|\Delta z|}{|\alpha|} = \frac{|\Delta z|}{|\xi - z|} < \frac{\frac{1}{3}d}{d} = \frac{1}{3}.$$

Consequently

$$|\varphi| \leq \frac{|\Delta z|^2}{|\alpha|^{n+2}}\sum_{k\geq 0}\frac{(k+n+1)!}{(k+2)!(n-1)!}\frac{|\Delta z^k|}{|\alpha^k|} < \frac{|\Delta z|^2}{|d|^{n+2}}\sum_{k\geq 0}\frac{(k+n+1)!}{(k+2)!(n-1)!}\frac{1}{3^k}.$$

Furthermore, we know that

$$\sum_{k\geq 0}\frac{(k+n+1)!}{(k+2)!(n-1)!}\frac{1}{3^k}$$

converges to some N. Hence $|\varphi| < \dfrac{|\Delta z|^2}{|d|^{n+2}} N = |\Delta z|^2 N(n,d)$. Taking the modulus of (19.3) we have

$$
\begin{aligned}
&\left| \frac{(n-1)!}{2\pi i \Delta z} \int_C \left[\frac{1}{(\xi - z - \Delta z)^n} - \frac{1}{(\xi - x)^n} - \frac{\Delta z}{(\xi - x)^{n+1}} \right] f(\xi) d\xi \right| \\
&\leq \frac{(n-1)!}{2\pi \, |\Delta z|} \int_C |\varphi| \, |f(\xi)| \, |d\xi| < \frac{(n-1)!}{2\pi \, |\Delta z|} \int_C |\Delta z|^2 N(n,d) \, |f(\xi)| \, |d\xi| \\
&= \frac{(n-1)!}{2\pi} |\Delta z| N(n,d) \int_C |f(\xi)| \, |d\xi| \, . \qquad (19.4)
\end{aligned}
$$

Notice, as we take Δz to 0, (19.4) also goes to 0. Therefore (19.2) is proven for n. ☐

Corollary 19.3 (Morera's Theorem). *Let f be continuous on a simply connected domain E and*

$$\int_C f(z) dz = 0$$

for all $C \subset E$. Then f is analytic on E.

The proof is left as an exercise.

Corollary 19.4 (Liouville's Theorem). *Let f be analytic on $\mathbb{C}$ and* $\sup_{z \in \mathbb{C}} |f(z)| = M < \infty$*. Then $f(z)$ is constant.*

The proof is left as an exercise.

Exercises

Exercise 19.1 Prove that

$$\sum_{k \geq 0} \frac{(k+n+1)!}{(k+2)!(n-1)!} \frac{1}{3^k}$$

converges.

Exercise 19.2 Prove Corollary 19.3.

Exercise 19.3 Prove Corollary 19.4.

LECTURE 20

Stronger Version of Morera's Theorem; Fundamental Theorem of Algebra

Recall the exact statement of the Cauchy Theorem (Theorem 15.5). If f is analytic on a simply connected region E, then

$$\int_C f(z)dz = 0 \qquad \forall\, C \subset E.$$

One can easily see that Morera's Theorem (Corollary 19.3) is a converse of the Cauchy Theorem, and we can state

THEOREM 20.1. *Let f be continuous on a simply connected region E. Then*

$$f \text{ is analytic on } E \iff \int_C f(z)dz = 0$$

for all simply connected closed contours $C \subset E$.

REMARK 20.2. *Theorem 20.1 is fundamental in Complex Analysis but does not appear to have a name. It can also be used for a new (equivalent) definition of an analytic function.*

1. A Stronger Version of Morera's Theorem

REMARK 20.3. *Morera's Theorem (and hence Theorem 20.1) can also be stated for multiply connected regions since each such region can be split into a union of simply connected regions. Thus, such a statement is unnecessary.*

The conditions of Morera's Theorem can be relaxed. Namely, the following theorem will be important in the next lecture.

THEOREM 20.4 (a stronger version of Morera's Theorem). *Let f be continuous on a simply connected region E. If $\int_T f(z)dz = 0$ for any triangle $T \subset E$ then f is analytic on E.*

PROOF. Without loss of generality we may assume that E is a disk (do you see why?) $\mathbb{D}_R(z_0)$. Let

$$F(z) := \int_{[z_0,z]} f(\xi)d\xi \ , \ z \in \mathbb{D}_R(z_0),$$

where $[z_0, z]$ is the segment connecting z_0 and z. Now consider (as we did in Theorem 16.4)

$$F(z + \Delta z) - F(z) = \int_{[z_0,z+\Delta z]} f(\xi)d\xi - \int_{[z_0,z]} f(\xi)d\xi. \tag{20.1}$$

Let T be the triangle with vertices $z_0, z, z+\Delta z$ (without any specific orientation). By the condition

$$0 = \int_T f(\xi)d\xi = \int_{[z_0,z]} f(\xi)d\xi + \int_{[z,z+\Delta z]} f(\xi)d\xi - \int_{[z_0,z+\Delta z]} f(\xi)d\xi.$$

Thus,

$$\int_{[z_0,z+\Delta z]} f(\xi)d\xi = \int_{[z_0,z]} f(\xi)d\xi + \int_{[z,z+\Delta z]} f(\xi)d\xi. \tag{20.2}$$

Substituting (20.2) into (20.1) yields

$$F(z+\Delta z) - F(z) = \int_z^{z+\Delta z} f(\xi)d\xi. \tag{20.3}$$

Consider now

$$\begin{aligned}\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) &= \frac{F(z+\Delta z)-F(z)-f(z)\Delta z}{\Delta z} \\ &\overset{(20.3)}{=} \frac{1}{\Delta z}\left(\int_z^{z+\Delta z} (f(\xi)-f(z))d\xi\right).\end{aligned}$$

Then by the triangle inequality

$$\begin{aligned}\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| &\le \frac{1}{|\Delta z|}\int_z^{z+\Delta z} |f(\xi)-f(z)||d\xi| \\ &\le \frac{1}{|\Delta z|}\max_{\xi\in[z,z+\Delta z]} |f(\xi)-f(z)|\cdot|\Delta z| \\ &= \max_{\xi\in[z,z+\Delta z]} |f(\xi)-f(z)| \to 0 \text{ as } \Delta z \to 0\end{aligned}$$

since f is continuous. Thus, $F(z)$ is analytic on E and $F'(z) = f(z)$. Moreover, by Theorem 19.2, $f(z)$ is also analytic on E. (recall Exercise 19.2). □

Remark 20.5. *Theorem 20.4 lets us replace arbitrary contours C with arbitrary triangles T. This will be crucial in the next lecture.*

2. The Fundamental Theorem of Algebra

We will now discuss some important corollaries of the Liouville Theorem. First of all (although we have previously shown this) the Liouville Theorem immediately implies that $|\sin z|$ is unbounded. Note that $|\sin x| \le 1$ in Real Analysis.

But perhaps the most striking corollary is the following famous statement.

Theorem 20.6 (Fundamental Theorem of Algebra). *Every polynomial of order $n \ge 1$ has exactly n zeros (counting multiplicity).*

Proof. We will first show that a polynomial

$$p_n(z) = \sum_{k=n}^{0} a_k z^k,\ n \in \mathbb{N}$$

has at least one zero. Assume to the contrary that $p_n(z)$ has no zeros.

Thus, $\min_{z\in\mathbb{C}} |p_n(z)| = m > 0$. Moreover, $f(z) := \dfrac{1}{p_n(z)}$ is analytic on $\mathbb{C}$ and

$$\max_{z\in\mathbb{C}} |f(z)| = \left(\min_{z\in\mathbb{C}} |p_n(z)|\right)^{-1} = \frac{1}{m} < \infty.$$

By the Liouville Theorem $f(z)$ is then constant. Hence, so is $p_n(z)$. Which is a contradiction. Therefore, $p_n(z)$ has at least one zero. Call it z_0. It follows that $p_n(z_0) = 0$ and

$$p_n(z) = p_n(z) - p_n(z_0) = \sum_{k=n}^{1} a_k(z^k - z_0^k). \tag{20.4}$$

Recalling the formula $(k = 1, 2, ..., n)$

$$z^k - z_0^k = (z - z_0)(z^{k-1} + z^{k-2}z_0 + \cdots + z_0^{k-1}),$$

(20.4) yields that

$$p_n(z) = (z - z_0)q_{n-1}(z)$$

with some polynomial of order $n - 1$. Thus, by induction we can now recover all n zeros of $p_n(z)$. $\square$

It is a note of interest that before the advent of Complex Analysis, the Fundamental Theorem of Algebra was the topic of the doctoral thesis completed by Gauss in 1798. Needless to say, Complex Analysis has simplified this proof.

Exercises

Exercise 20.1

(1) Show (differently from what you did before) that $\forall\ n \in \mathbb{Z}$

$$\frac{1}{2\pi i}\int_{C_1(0)} z^n dz = \begin{cases} 1 & \text{if } n = -1 \\ 0 & \text{if } n \neq -1. \end{cases}$$

(2) Show that

$$\frac{1}{2\pi i}\int_{C_1(0)} \left(z + \frac{1}{z}\right)^n \frac{dz}{z} = \frac{2^{n-1}}{\pi}\int_0^{2\pi} \cos^n\theta d\theta.$$

(3) Use (1) and (2) to prove Willis' formula

$$\int_0^{2\pi} \sin^{2k}\theta d\theta = \frac{(2k)!\pi}{2^{2k-1}(k!)^2}.$$

Exercise 20.2 Evaluate $\displaystyle\int_{C_{10}(0)} \frac{\sin z}{(z-\pi)^3}dz$.

Exercise 20.3 Evaluate $\displaystyle\frac{1}{2\pi}\int_0^{2\pi} \exp\left(e^{int}\right) dt$, for $n \in \mathbb{Z}$.

LECTURE 21

The Goursat Theorem

So far, we have defined analytic functions as continuously differentiable functions. I.e., we have always verified that the derivative of a function not only exists, but is also continuous. In this lecture, we will prove that verifying the continuity condition is unnecessary.

To prove this, we will first need a well-known statement from real analysis adjusted here for our purposes.

Recall that if Ω is a set, then $\operatorname{diam}\Omega \stackrel{\text{def}}{=} \sup\limits_{x,x'\in\Omega} \operatorname{dist}(x,x')$.

LEMMA 21.1 (Cantor's Theorem). *Let $\{\Omega_n\}$ be a sequence of sets in $\mathbb{C}$ such that (a) each $\Omega_n = \overline{\Omega}_n$ for all n, (b) $\Omega_1 \supset \Omega_2 \supset \ldots$, and (c) $\operatorname{diam}\Omega_n \to 0$ as $n \to \infty$. Then*

$$\Omega := \bigcap_{n\geq 1} \Omega_n = \{z_0\}.$$

PROOF. For each n, pick up an arbitrary $z_n \in \Omega_n$. By (b), $z_n, z_m \in \Omega_n$ for all $m > n$, and hence

$$|z_n - z_m| \leq \operatorname{diam}\Omega_n \xrightarrow{by\ (c)} 0, \text{ as } n \to \infty.$$

Thus, $\{z_n\}$ is a Cauchy sequence, and therefore it converges. We define $z_0 := \lim\limits_{n\to\infty} z_n$. From this, we can see that $z_0 \in \Omega_n$ for all n, and therefore, $z_0 \in \Omega$.

It remains to show that z_0 is the only point in Ω. To that end, let $w \in \Omega$. Then $w \in \Omega_n$ for all n, and hence $|z_0 - w| \leq \operatorname{diam}\Omega_n$. By (c), we can conclude that $z_0 = w$. $\square$

THEOREM 21.2 (Goursat's Theorem). *If f is differentiable on E, then f is analytic on E.*

PROOF. Without loss of generality, we may assume E is an open disk. The stronger version of Morera's Theorem will be a crucial argument in our proof.

Let T be a triangle and $\Delta := \operatorname{Int} T$. Partition Δ as shown in Figure 1.

Note that T need not be isosceles. Denote $T_k := \partial\Delta_k$, and orient each triangle positively. We can see from Figure 1 that

$$\int_T f(z)\, dz = \sum_{k=1}^{4} \int_{T_k} f(z)\, dz.$$

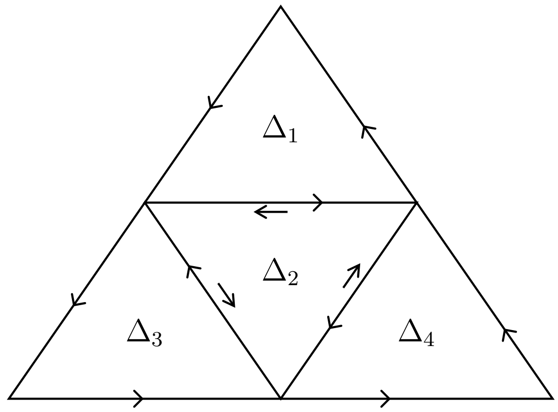


FIGURE 1. First partition of Δ into $\{\Delta_k\}_{k=1}^4$

Let $\Delta^{(1)}$ be one of the triangles $\{\Delta_k\}_{k=1}^4$ such that

$$\left|\int_{T^{(1)}} f(z)\,dz\right| = \max_{1\le k\le 4}\left\{\left|\int_{T_k} f(z)\,dz\right|\right\}.$$

We then have

$$\begin{cases} \left|\int_T f(z)\,dz\right| \le 4\left|\int_{T^{(1)}} f(z)\,dz\right| \\ |T^{(1)}| = \frac{1}{2}|T| \\ \Delta \supset \Delta^{(1)} \\ \operatorname{diam}\Delta^{(1)} = \frac{1}{2}\operatorname{diam}\Delta. \end{cases} \tag{21.1}$$

We can now repeat the same procedure with $\Delta^{(1)}$ and get $\Delta^{(2)}$, and so on. So for each n, we get

$$\left|\int_T f(z)\,dz\right| \le 4^n\left|\int_{T^{(n)}} f(z)\,dz\right| \tag{21.2a}$$

$$|T^{(n)}| = \left(\frac{1}{2}\right)^n |T| \tag{21.2b}$$

$$\Delta \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \ldots \supset \Delta^{(n)} \tag{21.2c}$$

$$\operatorname{diam}\Delta^{(n)} = \left(\frac{1}{2}\right)^n \operatorname{diam}\Delta. \tag{21.2d}$$

By Lemma 21.1,

$$\bigcap_{n\ge 1}\Delta^{(n)} = \{z_0\}.$$

By our assumption, f is differentiable at $z = z_0$. So we know that for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $z \in \mathbb{D}_\delta(z_0)$,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon|z - z_0|. \tag{21.3}$$

Now, choose n such that

$$\operatorname{diam}\Delta^{(n)} = \left(\frac{1}{2}\right)^n \operatorname{diam}\Delta < \delta.$$

We can see that $\Delta^{(n)} \subset \mathbb{D}_\delta(z_0)$. Since the functions 1 and z are analytic, then by Cauchy's Theorem,

$$\int_{T^{(n)}} dz = 0 = \int_{T^{(n)}} z\, dz.$$

Therefore

$$\left|\int_{T^{(n)}} f(z)\, dz\right| = \left|\int_{T^{(n)}} f(z) - f(z_0) - f'(z_0)(z - z_0)\, dz\right|.$$

By the triangle inequality, we get

$$\left|\int_{T^{(n)}} f(z)\, dz\right| \leq \int_{T^{(n)}} |f(z) - f(z_0) - f'(z_0)(z - z_0)|\; |dz|.$$

From equation (21.3), we get

$$\left|\int_{T^{(n)}} f(z)\, dz\right| \leq \varepsilon \int_{T^{(n)}} \underbrace{|z - z_0|}_{\leq \operatorname{diam} \Delta^{(n)}}\; |dz| \leq \varepsilon \operatorname{diam} \Delta^{(n)} \underbrace{\int_{T^{(n)}} |dz|}_{|T^{(n)}|}.$$

From (21.2d), we see that $\operatorname{diam} \Delta^n = \left(\frac{1}{2}\right)^n \operatorname{diam} \Delta$ and from (21.2b), we have $|T^{(n)}| \leq \left(\frac{1}{2}\right)^n |T|$. Therefore

$$\left|\int_{T^{(n)}} f(z)\, dz\right| \leq \varepsilon \left(\frac{1}{4}\right)^n \operatorname{diam} \Delta |T|.$$

Letting $C := |T| \operatorname{diam} \Delta$, a constant, we get

$$\left|\int_{T^{(n)}} f(z)\, dz\right| \leq \frac{\varepsilon}{4^n} C. \tag{21.4}$$

It follows from (21.2a) and (21.4) that

$$\left|\int_{T} f(z)\, dz\right| \leq 4^n \frac{\varepsilon}{4^n} C = \varepsilon C. \tag{21.5}$$

Since ε is arbitrary, (21.5) implies

$$\int_T f(z)\, dz = 0$$

for every $T \subset E$. By the stronger Morera's Theorem, f is analytic on E. □

COROLLARY 21.3. *If f is differentiable, then f' is continuous.*

This means that in the original definition of analytic functions we could have removed the extra condition of continuity of f'. But we did not have all the necessary tools then.

Exercises

Exercise 21.1 Show that if you drop any condition of (a), (b), or (c) in Lemma 21.1, then Cantor's Theorem fails.

Exercise 21.2 Consider the function $f(z) = \dfrac{1}{z^2}$. We know from Exercise 20.11 that

$$\int_C \frac{dz}{z^2} = 0. \tag{21.6}$$

for any contour C such that $0 \in \operatorname{Int} C$.

(a) Discuss why Morera's Theorem doesn't apply.

(b) Does (21.6) hold for every C? If not, provide a counterexample.

LECTURE 22

Taylor Series

In Lecture 4, we introduced the important concept of a convergent power series. We later proved that a convergent power series was an analytic function, and used this fact to define e^z, $\sin z$, $\cos z$, among others. In this lecture, we prove that the converse is also true. But first, an important proposition.

Recall, we say a sequence of functions $\{f_n\}$ defined on C converges uniformly to f if

$$\sup_{z\in C} |f_n(z) - f(z)| \to 0 \text{ as } n \to \infty. \tag{22.1}$$

and write $f_n \rightrightarrows f$.

Proposition 22.1 (on passing to the limit under the integral sign). *Let $\{f_n(z)\}$ be a sequence of continuous functions defined on a contour C of finite length. Then*

$$f_n \rightrightarrows f \implies \int_C f_n(z)\,dz \to \int_C f(z)\,dz, \quad n \to \infty$$

Proof. Observe:

$$\begin{aligned}\left|\int_C f_n(z)\,dz - \int_C f(z)\,dz\right| &= \left|\int_C (f_n(z) - f(z))\,\,dz\right| \le \int_C |f_n(z) - f(z)|\,|dz| \\ &\le \int_C \sup_{z\in C}|f_n(z) - f(z)|\,|dz| \le \sup_{z\in C}|f_n(z) - f(z)| \cdot |C|.\end{aligned}$$

which we can see goes to zero as n goes to infinity. ☐

Corollary 22.2. *Let C be a contour of finite length, and $\{f_n\}$ be a sequence of continuous functions on C. If $\sum_{n\ge 0} f_n(z)$ converges uniformly on C, then*

$$\int_C \sum_{n\ge 0} f_n(z)\,dz = \sum_{n\ge 0}\int_C f_n(z)\,dz. \tag{22.2}$$

Proof. Note that as $N \to \infty$, the sequence

$$S_N(z) = \sum_{n=0}^{N} f_n(z)\,dz$$

converges uniformly to $S(z) := \sum_{n\ge 0} f_n(z)$, and satisfies the conditions of Proposition 22.1, and thus

$$\lim_{N\to\infty}\int_C S_N(z)\,dz = \int_C S(z)\,dz. \tag{22.3}$$

On the other hand,

$$\lim_{N\to\infty}\int_C S_N(z)\,dz = \lim_{N\to\infty}\sum_{n=0}^{N}\int_C f_n(z)\,dz \stackrel{\text{def}}{=} \sum_{n\geq 0}\int_C f_n(z)\,dz. \tag{22.4}$$

Combining (22.3) and (22.4) yields (22.2). □

Remark 22.3. *Proposition 22.1 and Corollary 22.2 actually say when we can pass to the limit under the integral sign and when we can integrate series term-wise. That is*

$$\lim_{n\to\infty}\int_C f_n(z)\,dz = \int_C \lim_{n\to\infty} f_n(z)\,dz, \quad \int_C \sum_{n\geq 0} f(z)\,dz = \sum_{n\geq 0}\int_C f(z)\,dz.$$

Theorem 22.4 (Taylor's theorem). *Every function which is analytic on a disk $\mathbb{D}_R(z_0)$ can be* <u>*uniquely*</u> *expanded into the Taylor series*

$$f(z) = \sum_{n\geq 0} a_n (z-z_0)^n, \tag{22.5}$$

where

$$a_n = \frac{1}{n!} f^{(n)}(z_0), \tag{22.6}$$

and the series (22.5) *is absolutely convergent on $\mathbb{D}_R(z_0)$.*

Proof. Consider $g(z) := f(z+z_0)$. Notice that g will be analytic on $\mathbb{D}_R(0)$ and by Cauchy's Theorem,

$$g(z) = \frac{1}{2\pi i}\int_{C_\rho(0)} \frac{g(\xi)}{\xi - z}\,d\xi, \quad \forall z \in \mathbb{D}_\rho(0), \quad \text{for } \rho < R \tag{22.7}$$

Notice that $\left|\frac{z}{\xi}\right| < 1$, and transform $\frac{1}{\xi - z}$ as follows:

$$\frac{1}{\xi - z} = \frac{1}{\xi}\cdot\frac{1}{1-\frac{z}{\xi}} = \frac{1}{\xi}\sum_{n\geq 0}\left(\frac{z}{\xi}\right)^n$$

But we know $\sum_{n\geq 0}\left(\frac{z}{\xi}\right)^n$ uniformly converges to $\frac{1}{1-\xi/z}$, and hence

$$\frac{1}{2\pi i}\int_{C_\rho(0)} \frac{g(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\int_{C_\rho(0)} \sum_{n\geq 0}\frac{g(\xi)}{\xi}\left(\frac{z}{\xi}\right)^n\,d\xi$$

$$\stackrel{\text{by Cor. 22.2}}{=} \sum_{n\geq 0}\underbrace{\left(\frac{1}{2\pi i}\int_{C_\rho(0)}\frac{g(\xi)}{\xi^{n+1}}\,d\xi\right)}_{=\frac{g^{(n)}(0)}{n!}} z^n = \sum_{n\geq 0} a_n z^n, \quad a_n := \frac{g^{(n)}(0)}{n!}.$$

Thus,

$$g(z) = \sum_{n\geq 0} a_n z^n, \qquad a_n = \frac{g^{(n)}(0)}{n!}. \tag{22.8}$$

Recall that $f(z) = g(z - z_0)$ and observe that $g^{(n)}(0) = f^{(n)}(z_0)$. With this in mind, we substitute $z - z_0$ for z in (22.8), and we arrive at (22.5) and (22.6). We leave the proof of uniqueness as an exercise. $\square$

From this, we see that if a function f is analytic on some $E \subseteq \mathbb{C}$, then the power series of f will converge on $\mathbb{D}_R(z_0)$, where R is chosen such that $\mathbb{D}_R(z_0) \subseteq E$, for all $z_0 \in E$.

Recall Theorem 6.3, in which we proved that if $\sum_{n\geq 0} a_n(z - z_0)^n$ converges on $\mathbb{D}_R(z_0)$, then $f(z) := \sum_{n\geq 0} a_n(z - z_0)^n$ is analytic on $\mathbb{D}_R(z_0)$. This combined with Taylor's Theorem gives us:

THEOREM 22.5. *A function f is analytic on $\mathbb{D}_R(z_0)$ if and only if*

$$f(z) = \sum_{n\geq 0} a_n(z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

converges on $\mathbb{D}_R(z_0)$.

EXAMPLE 22.6. *Consider the function $f(z) = \dfrac{1}{1-z}$. This function is analytic everywhere except at $z = 1$. So we can only consider its Taylor series expansion on $\mathbb{D}$ since otherwise 1 would get inside the disk and then f would not be analytic inside such a disk. Hence*

$$\forall\ z \in \mathbb{D} \qquad \frac{1}{1-z} = \sum_{n\geq 0} z^n.$$

Exercises

Exercise 22.1 Prove the expansion in (22.5) is unique.

Exercise 22.2 Show that

$$\log(1 + z) = \sum_{n\geq 1} (-1)^{n+1} \frac{z^n}{n},$$

and find where the series converges.

LECTURE 23

Laurent Series

As we know, Taylor series are very common in calculus. The main object of this lecture, the Laurent series, is not typically considered in calculus but is a central object of complex analysis.

DEFINITION 23.1. *A series of the form*

$$\sum_{n=-\infty}^{\infty} a_n (z - z_o)^n \tag{23.1}$$

is called a <u>*Laurent series*</u>.

Note that n in (23.1) is any integer.

PROPOSITION 23.2. *The domain of convergence of*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n \tag{23.2}$$

is the annulus

$$R_1 < |z - z_0| < R_2 \tag{23.3}$$

where

$$R_1 = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|},$$
$$R_2 = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}.$$

We also have that $f(z)$ is analytic on the same annulus.

PROOF. Without loss of generality we may set $z_0 = 0$. Split (23.2) as follows:

$$f(z) = \underbrace{\sum_{n\le -1} a_n z^n}_{f_1(z)} + \underbrace{\sum_{n\ge 0} a_n z^n}_{f_2(z)}.$$

The series $f_2(z)$ is a Taylor series and hence its domain of convergence is the disk $\mathbb{D}_{R_2}(0)$, where by the Cauchy-Hadamard formula

$$R_2 = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1}.$$

Turn then to f_1 where we have

$$f_1(z) = \sum_{n\geq 1} a_{-n}\left(\frac{1}{z}\right)^n$$

which is a Taylor series with respect to $\frac{1}{z}$ which converges on $\left\{z : |\frac{1}{z}| < r\right\}$ where

$$r = \left(\limsup_{n\to\infty} \sqrt[n]{|a_{-n}|}\right)^{-1}.$$

But $\left\{z : |\frac{1}{z}| < r\right\} = \left\{z : |z| > \frac{1}{r}\right\}$, i.e. denoting $R_1 := \frac{1}{r}$ we conclude that the domain of convergence of f_2 is therefore $\mathbb{C} \setminus \overline{\mathbb{D}}_{R_1}(0)$ and f_1 and f_2 both converge on $\mathbb{D}_{R_2} \bigcap \left(\mathbb{C} \setminus \overline{\mathbb{D}}_{R_1}(0)\right)$ which is the annulus $\{z \mid R_1 < |z| < R_2\}$. Notice that both f_1 and f_2 are analytic because they are Taylor series. Hence their sum, $f(z)$ is also analytic on the region where they are both analytic. $\square$

Remark 23.3. *If $a_n = 0$ for all $n \leq -1$, then the Laurent series* (23.2) *becomes a Taylor series. The only condition on R_1 and R_2 in Proposition 23.2 is $R_1 < R_2$. Moreover, R_1 could be 0 and R_2 could be ∞.*

Example 23.4. *The series $\sum_{n\geq -1} z^n, \sum_{n\geq 0} \frac{1}{n! z^n}$, and $\sum_{n\geq 1} z^{1-n}$ are all Laurent series.*

And now the main theorem of this lecture.

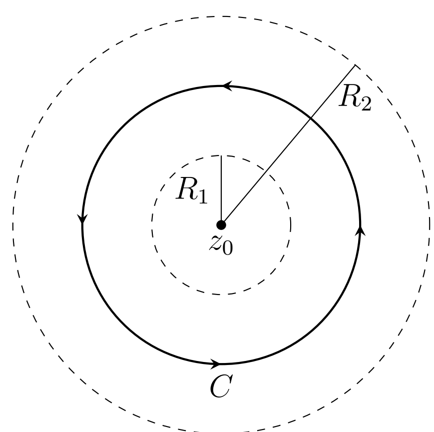


Figure 1. The contour C in Laurent's Theorem

Theorem 23.5 (Laurent's Theorem). *Any function $f(z)$ that is analytic on an annulus*

$$R_1 < |z - z_0| < R_2$$

can be expanded into the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{23.4}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi, \qquad n \in \mathbb{Z},$$

where C is any contour like in Figure 1.

Proof. As in the proof of the Taylor Theorem, we introduce

$$g(z) = f(z+z_0), \qquad R_1 < |z| < R_2.$$

By the Cauchy formula

$$g(z) = \frac{1}{2\pi i}\int_{C_{\rho_2}\bigcup(-C_{\rho_1})} \frac{g(\xi)}{\xi - z}d\xi = \underbrace{\frac{1}{2\pi i}\int_{C_{\rho_2}} \frac{g(\xi)}{\xi - z}d\xi}_{g_2(z)} + \underbrace{\frac{1}{2\pi i}\int_{-C_{\rho_1}} \frac{g(\xi)}{\xi - z}d\xi}_{g_1(z)} \tag{23.5}$$

where C_{ρ_1} and C_{ρ_2} are like in Figure 2.

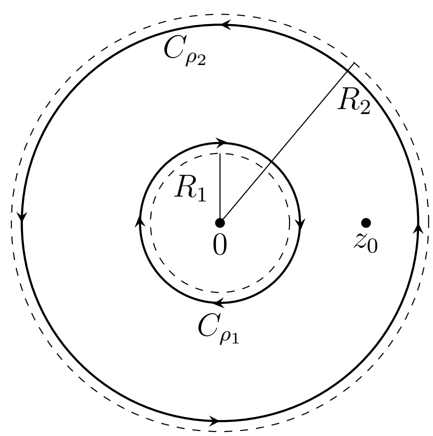


Figure 2. The contour $C_{\rho_2} \cup (-C_{\rho_1})$

As we did in the proof of Taylor's formula, consider

$$\frac{1}{\xi - z} = \frac{1}{\xi}\cdot\frac{1}{1-\frac{z}{\xi}} = \frac{1}{\xi}\sum_{n\geq 0}\left(\frac{z}{\xi}\right)^n. \tag{23.6}$$

If $\xi \in C_{\rho_2}$ then $\left|\frac{z}{\xi}\right| < 1$ and (23.6) uniformly converges on $\operatorname{Int} C_{\rho_2}$ and hence

$$g_2(z) = \frac{1}{2\pi i}\int_{C_{\rho_2}} \frac{g(\xi)}{\xi}\sum_{n\geq 0}\left(\frac{z}{\xi}\right)^n d\xi = \sum_{n\geq 0} a_n z^n, \tag{23.7}$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_{\rho_2}} \frac{g(\xi)}{\xi^{n+1}}d\xi = \frac{g^{(n)}(0)}{n!}, n \geq 0.$$

If $\xi \in C_{\rho_1}$ then $\left|\frac{\xi}{z}\right| < 1$ and

$$\frac{1}{\xi - z} = -\frac{1}{z} \cdot \frac{1}{1 - \frac{\xi}{z}} = -\frac{1}{z} \sum_{n \geq 0} \left(\frac{\xi}{z}\right)^n$$

converges uniformly on $\operatorname{Ext} C_{\rho_1} (\operatorname{Ext} C \stackrel{\text{def}}{=} \mathbb{D} \setminus \overline{\operatorname{Int} C})$ and hence

$$\begin{aligned}
g_1(z) &= \frac{1}{2\pi i} \int_{-C_{\rho_1}} g(\xi) \left(-\frac{1}{z}\right) \sum_{n \geq 0} \left(\frac{\xi}{z}\right)^n d\xi \\
&= -\sum_{n \geq 0} \left(\frac{1}{2\pi i} \int_{-C_{\rho_1}} g(\xi) \xi^n d\xi\right) z^{-(n+1)} \\
&\stackrel{\text{reindexing}}{=} \sum_{n \geq 1} \left(\frac{1}{2\pi i} \int_{C_{\rho_1}} \frac{g(\xi)}{\xi^{-n+1}} d\xi\right) z^{-n}.
\end{aligned} \tag{23.8}$$

Plugging (23.7) and (23.8) into (23.5) we have

$$g(z) = g_1(z) + g_2(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \begin{cases} \displaystyle\int_{C_{\rho_1}} \frac{g(\xi)}{\xi^{n+1}} d\xi, & n \leq -1 \\ \displaystyle\int_{C_{\rho_2}} \frac{g(\xi)}{\xi^{n+1}} d\xi, & n \geq 0 \end{cases}.$$

But $C_{\rho_1} \sim C_{\rho_2} \sim C$ where C is an contour in $R_1 < |z| < R_2$ enclosing 0 and hence

$$a_n = \frac{1}{2\pi i} \int_C \frac{g(\xi)}{\xi^{n+1}} d\xi, \quad n \in \mathbb{Z}.$$

Therefore the theorem is proven for $z_0 = 0$ and hence by the shifting argument it's also proven for any z_0. □

Remark 23.6. *The Laurent representation of an analytic function is not unique and depends on the choice of the annulus. For example,*

$$\begin{aligned}
\frac{1}{1-z} &= \sum_{n \geq 0} z^n, \qquad 0 < |z| < 1 \\
\frac{1}{1-z} &= -\sum_{n \geq 1} z^{-n}, \qquad 1 < |z| < \infty.
\end{aligned}$$

Exercises

Exercise 23.1 Find the domains of convergence of the series in Example 23.4 and find the functions to which these series converge.

Exercise 23.2 Use known Taylor series to derive the Laurent series of the given functions in the indicated annuli.

(1) $z + \frac{1}{z}, \quad |z-1| > 1$

(2) $\frac{\sin(1/z)\cos(1/z)}{z}, \quad |z| > 0$

(3) $\frac{1}{(3z-1)(2z+1)}, \quad \frac{1}{3} < |z| < \frac{1}{2}$

Exercise 23.3 Evaluate

$$\int_{C_1(0)} e^{z^2+\frac{1}{z}}\,dz$$

by using an appropriate Laurent series.

LECTURE 24

Singularities of Analytic Functions

Recall the notation: $\mathring{\mathbb{D}}_R(z_0) = \mathbb{D}_R(z_0) \backslash \{z_0\}$.

Definition 24.1. *A point z_0 is called an* <u>*isolated singularity*</u> *of an analytic function $f(z)$ if there exists some $R > 0$ such that f is analytic on $\mathring{\mathbb{D}}_R(z_0)$.*

In other words, f is analytic in <u>some neighborhood</u> of z_0 but <u>not</u> at z_0 itself. E.g. $z = 0$ is an isolated singularity of $1/z$. In some sources, any point z_0 at which a function fails to be analytic is called singular. E.g. if we define $f(z) = \sqrt{z}$ with the branch cut along $\mathbb{R}^+$, then $f(z)$ fails to be analytic for any $z \in \mathbb{R}^+$. However, such singularities are not isolated. There is a huge difference between isolated and not isolated singularities. The former are much more important which will be demonstrated throughout the rest of the course. For this reason, we reserve the right to refer to isolated singularities as just singularities.

If a function $f(z)$ has an isolated singularity at z_0, then by the Laurent Theorem (Theorem 23.5) it can be represented on some annulus $\mathring{\mathbb{D}}_R(z_0)$ as

$$f(z) = \underbrace{\sum_{n \geq 1} a_{-n}(z - z_0)^{-n}}_{f_1(z)} + \sum_{n \geq 0} a_n (z - z_0)^n. \tag{24.1}$$

The following definition is <u>very important</u>.

Definition 24.2. *Let z_0 be an isolated singularity of an analytic function $f(z)$.*

(1) *If $f_1(z) = 0$ then z_0 is called a* <u>*removable singularity*</u>.

(2) *If $f_1(z) = \sum_{n=1}^{N} a_{-n} z^{-n}$, where $N < \infty$ and $a_{-N} \neq 0$ then z_0 is called a* <u>*pole of order N*</u>.

(3) *If $f_1(z)$ has infinitely many nonzero terms then z_0 is called an* <u>*essential singularity*</u>.

Example 24.3. *Show that*

(1) *$f(z) = \dfrac{\sin z}{z}$ has a removable singularity at $z = 0$.*

$$f(z) = \frac{1}{z} \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n \geq 0} \frac{(-1)^n z^{2n}}{(2n+1)!}.$$

Thus $f_1(z) = 0$, and by definition, this implies that $z = 0$ is a removable singularity.

(2) $f(z) = \dfrac{\sin z}{z^3}$ *has an order 2 pole at* $z = 0$.

$$f(z) = \frac{1}{z^3}\sum_{n\geq 0}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n\geq 0}\frac{(-1)^n z^{2n-2}}{(2n+1)!} = \frac{1}{z^2} + \sum_{n\geq 0}\frac{(-1)^{n+1} z^{2n}}{(2n+3)!}.$$

Thus $f_1(z) = \dfrac{1}{z^2}$ *is a finite sum with* $N = 2$, *and by definition, this implies that* $z = 0$ *is a pole of order 2.*

(3) $f(z) = e^{1/z}$ *has an essential singularity at* $z = 0$.

$$f(z) = \sum_{n\geq 0}\frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n\geq 1}\frac{1}{n!}z^{-n} + 1.$$

Thus $f_1(z)$ *is an infinite sum, and by definition this implies that* $z = 0$ *is an essential singularity.*

PROPOSITION 24.4. *Let* z_0 *be an isolated singularity of* $f(z)$. *Then*

(1) $\lim_{z\to z_0} f(z)$ *exists if and only if* z_0 *is removable.*

(2) *If for some* $N \in \mathbb{N}$, *there exists* $\lim_{z\to z_0}(z-z_0)^N f(z) \neq 0$, *and if for all* $n > N$ $\lim_{z\to z_0}(z-z_0)^n f(z) = 0$, *then* z_0 *is a pole of order* N.

The proof is left as an exercise.

REMARK 24.5. *If* z_0 *is a removable singularity of* $f(z)$, *then the 'hole'* z_0 *can be 'patched' by assuming that* $f(z_0) = \lim_{z\to z_0} f(z)$. *It immediately follows from* (24.1) *that*

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0}\sum_{n\geq 0} a_n(z-z_0)^n \overset{\text{unif conv}}{=} \sum_{n\geq 0} a_n \lim_{z\to z_0}(z-z_0)^n = a_0.$$

I.e. the function

$$\tilde{f}(z) := \begin{cases} f(z) & , \ z \neq z_0 \\ a_0 & , \ z = z_0 \end{cases}$$

is analytic on $\mathbb{D}_R(z_0)$. *I.e. indeed, the 'patched' function* $\tilde{f}$ *becomes analytic at* z_0. *In the future we will make no distinction between* f *and* $\tilde{f}$. *E.g. we will call the function* $f(z) = \dfrac{\sin z}{z}$ *analytic on* $\mathbb{C}$. *Thus, removable singularities are 'innocent' and don't cause any problems.*

Poles on the contrary cannot be fixed this way as the following proposition suggests.

PROPOSITION 24.6. *If* z_0 *is a pole then* $\lim_{z\to z_0}|f(z)| = \infty$.

The proof is left as an exercise.

PROPOSITION 24.7. *If* $f(z)$ *is analytic on* $\mathring{\mathbb{D}}_R(z_0)$ *and* $\sup_{z\in\mathring{\mathbb{D}}_R(z_0)}|f(z)| \leq M < \infty$, *then* z_0 *is a removable singularity.*

The proof is left as an exercise.

PROPOSITION 24.8. *If z_0 is a removable singularity of $f(z)$ then*

$$\exists\, \mathbb{D}_R(z_0): \quad f(z) = (z - z_0)^m \varphi(z)$$

with some $m \in \mathbb{N}_0$; $\varphi(z)$ is analytic on $\mathbb{D}_R(z_0)$ and $\varphi(z_0) \neq 0$.

The proof is left as an exercise.

Essential singularities are the worst of the three. As the important theorem below shows, $\lim\limits_{z \to z_0} |f(z)|$ does not exist.

THEOREM 24.9 (Casorati-Weierstrass Theorem). *If z_0 is an essential singularity of $f(z)$, then*

$$\overline{f\left(\mathring{\mathbb{D}}_\delta(z_0)\right)} = \mathbb{C} \qquad \forall\, \delta > 0,$$

that is the closure of the image of any punctured δ-neighborhood of z_0 under f is the whole complex plane.

PROOF. By contradiction. Assume that there exists a $c \in \mathbb{C}$ such that

$$\exists\, \varepsilon, \delta > 0: \quad \forall\, z \in \mathring{\mathbb{D}}_\delta(z_0): \quad |f(z) - c| \geq \varepsilon > 0. \tag{24.2}$$

Consider $\psi(z) = \dfrac{1}{f(z) - c}$. By (24.2), $\psi(z)$ is bounded on $\mathring{\mathbb{D}}_\delta(z_0)$ and hence, by Proposition 24.7, z_0 is a removable singularity of ψ. Then by Proposition 24.8

$$\psi(z) = (z - z_0)^m \varphi(z)\ ,\ \varphi(z_0) \neq 0.$$

Thus

$$f(z) = (z - z_0)^{-m} \tilde{\varphi}(z) + c \tag{24.3}$$

where $\tilde{\varphi}(z) = \dfrac{1}{\varphi(z)}$ is bounded on $\mathbb{D}_\delta(z_0)$ for δ small enough.

But (24.3) means that z_0 is either removable or a pole! We have reached a contradiction. $\square$

REMARK 24.10. *Using $f(z) = e^{1/z}$ and $z = 0$, we can see how $f\left(\mathring{\mathbb{D}}_\delta(0)\right) = \mathbb{C} \setminus \{0\}$. I.e. f takes on all values in the complex plane except for 0.*

A stronger version of this statement can be found in the following theorem which we present without proof.

THEOREM 24.11 (Picard's Theorem). *If f has an essential singularity at z_0 then $f(z) = c$ has infinitely many solutions in any punctured neighborhood of z_0 and for any value of c in $\mathbb{C}$ except perhaps one.*

Exercises

Exercise 24.1 Prove Proposition 24.4

Exercise 24.2 Prove Proposition 24.6

Exercise 24.3 Prove Proposition 24.7

Exercise 24.4 Prove Proposition 24.8

Exercise 24.5 Verify the Picard Theorem for $f(z) = e^{\frac{1}{z}}$. Namely, you must show that $f\left(\mathring{\mathbb{D}}_\delta(0)\right) = \mathbb{C}\setminus\{0\} \quad \forall\delta > 0$ and $f(z)$ takes on every value at infinitely many points $z_n \in \mathring{\mathbb{D}}_\delta(0),\ \forall\delta > 0$.

LECTURE 25

Singularities at Infinity

As we know, infinity is not a specific point, but sometimes it is convenient to treat infinity as a specific point. One of the ways to do it follows from the picture below.

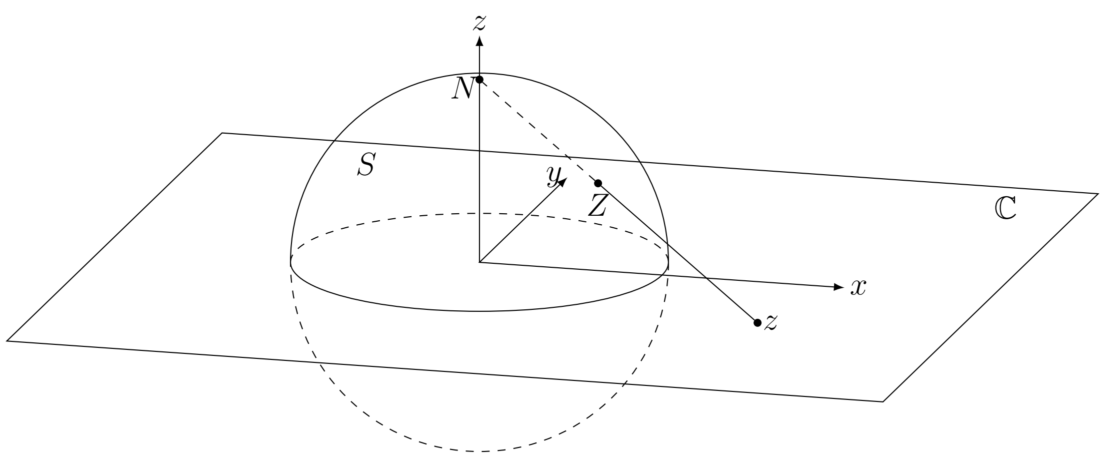


FIGURE 1. Stereographic projection onto the complex plane

We can identify any $z \in \mathbb{C}$ with a unique point $Z \in S$ where

$$S := \left\{(x, y, z) : x^2 + y^2 + z^2 = 1\right\}.$$

For each point $z \in \mathbb{C}$ we associate a point in S by drawing a straight line throught the North Pole, N, and z. The northern hemisphere can then be associated with $|z| > 1$ and the southern hemisphere can be associated with $|z| < 1$.

It is geometrically clear that if $Z \to N$ then $|z| \to \infty$. The one-to-one correspondence between $S \backslash \{N\}$ and $\mathbb{C}$ can be extended to S by letting N map to ∞. This correspondence is called stereographic projection and is denoted by P. One can then define complex infinity by $\infty = P(N)$.

But for our purposes we will use a simpler way. Consider the function $1/z$ which is analytic on $\mathbb{C} \backslash \{0\}$. Now set

$$\infty \overset{\text{def}}{:=} \lim_{z \to 0} \frac{1}{z}. \tag{25.1}$$

Let $f(z)$ be analytic on some $|z| > R$. Then by the Laurent Theorem

$$f(z) = \sum_{n \geq 1} a_{-n} z^{-n} + \underbrace{\sum_{n \geq 0} a_n z^n}_{f_2(z)}. \tag{25.2}$$

One notes that if f_2 is not constant then the function $f(z)$ 'blows up' as $|z| \to \infty$, and we can think of $z = \infty$ as a singular point of $f(z)$. With this approach in hand, we can classify ∞ as an 'isolated' singularity the same way we did for 'finite' isolated singularities.

However, (25.1) prompts a simpler definition.

DEFINITION 25.1. *Let $f(z)$ be analytic on $|z| > R$ and $g(z) := f\left(\frac{1}{z}\right)$. Then $z = \infty$ is called:*

1. *a removable singularity at infinity of $f(z)$ if $z = 0$ is a removable singularity of $g(z)$.*
2. *a pole of order N at infinity of $f(z)$ if $z = 0$ is a pole of order N of $g(z)$.*
3. *an essential singularity at infinity of $f(z)$ if $z = 0$ is an essential singularity of $g(z)$.*

LEMMA 25.2. *Assume that z_0 is an essential singularity of $f(z)$. Then z_0 is an essential singularity of $h(z) = 1/f(z)$.*

PROOF. Assume that z_0 is a removable singularity of $h(z)$. Thus, $\lim\limits_{z\to z_0} h(z)$ exists. If $\lim\limits_{z\to z_0} h(z) \neq 0$, then $\lim\limits_{z\to z_0} f(z)$ exists. But then z_0 is a removable singularity in $f(z)$ (by Proposition 24.4), a contradiction. If $\lim\limits_{z\to z_0} h(z) = 0$, then $\lim\limits_{z\to z_0} |f(z)| = \infty$. But then z_0 is a pole (by Proposition 24.6), a contradiction. Now assume that z_0 is a pole of $h(z)$. Then $\lim\limits_{z\to z_0} |h(z)| = \infty$ (by Proposition 24.6). If follows that $\lim\limits_{z\to z_0} |f(z)| = 0 \Rightarrow \lim\limits_{z\to z_0} f(z) = 0$. But then z_0 is a removable singularity in $f(z)$ (by Propostion 24.4), a contradiction. Thus, by reductio ad absurdum z_0 is an essential singularity of $h(z)$. $\square$

EXAMPLE 25.3.

1. *$1/z$ has a removable singularity at ∞.*
2. *a polynomial $p_n(z) = \sum\limits_{k=n}^{1} a_k z^k$, where $a_n \neq 0$, has an order n pole at ∞.*
3. *e^z has an essential singularity at ∞.*
4. *$e^{1/z}$ has a removable singularity at ∞.*

Exercises

Exercise 25.1 Give a classification of singularities at ∞ similar to Definition 24.2 using representation (25.2). Show it is equivalent to Definition 25.1. Try to make your statements as concise as possible.

Exercise 25.2 State and prove an analog of Proposition 24.7 for a removable singularity at ∞.

Exercise 25.3 State and prove an analog of Proposition 24.8 for a removable singularity at ∞.

Exercise 25.4 Describe all singularities (including ∞) of

(1) $\dfrac{z}{z^2+1} - \dfrac{1}{z}$

(2) $\dfrac{e^z}{e^z - 1}$

(3) $\sinh z$

LECTURE 26

The Residue

In this lecture we finally get to the concept of residue.

1. Residue at an isolated singularity

DEFINITION 26.1. *Let z_0 be an isolated singularity of f and C be a simply connected contour enclosing z_0 and laying in the domain of analyticity of f. Then*

$$\frac{1}{2\pi i}\int_C f(z)dz =: \operatorname{Res}\{f(z), z_0\} \tag{26.1}$$

is called the residue *of f at z_0. The residue can also be denoted as* $\operatorname{Res}(f, z_0)$, *or* $\operatorname*{Res}_{z=z_0} f(z)$.

REMARK 26.2. *The contour C in (26.1) can of course be deformed the way we like and Definition 26.1 doesn't depend on C. Thus the residue is determined by the function. What is remarkable about this concept is that the residue can be evaluated by certain simple formulas.*

THEOREM 26.3. *Let f be analytic on E and z_0 be an isolated singularity of f. Then*

(1)
$$\boxed{\operatorname{Res}(f, z_0) = a_{-1} \qquad (26.2)}$$
where a_{-1} is the corresponding coefficient in the Laurent expansion of $f(z)$ on some $\mathring{\mathbb{D}}_R(z_0)$.

(2) *If z_0 is a pole of order $N \in \mathbb{N}$ then*
$$\boxed{\operatorname{Res}(f, z_0) = \frac{1}{(N-1)!}\lim_{z\to z_0}\frac{d^{N-1}}{dz^{N-1}}\{(z-z_0)^N f(z)\}. \qquad (26.3)}$$

PROOF. Start with (1). By the Laurent Theorem,

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi, \qquad n \in \mathbb{Z}. \tag{26.4}$$

In particular, setting $n = -1$ in (26.4), we have

$$a_{-1} = \frac{1}{2\pi i}\int_C f(\xi)d\xi \overset{\text{def}}{=} \operatorname{Res}(f, z_0).$$

Derive now (2). If z_0 is a pole of order N then on $\mathring{\mathbb{D}}_R(z_0)$

$$
\begin{aligned}
f(z) &= \sum_{n\geq -N} a_n(z-z_0)^n = (z-z_0)^{-N}\sum_{n\geq -N} a_n(z-z_0)^{n+N} \\
&\overset{\text{reindexing}}{=} (z-z_0)^{-N}\underbrace{\sum_{n\geq 0} a_{n-N}(z-z_0)^n}_{\text{a Taylor series}}
\end{aligned}
$$

$$
\Rightarrow (z-z_0)^N f(z) = a_{-N} + \cdots + a_{-1}(z-z_0)^{N-1} + \underbrace{\sum_{n\geq N} a_{n-N}(z-z_0)^n}_{=:g(z)}. \tag{26.5}
$$

Differentiating (26.5) $N-1$ times, we have

$$
\frac{d^{N-1}}{dz^{N-1}}(z-z_0)^N f(z) = (N-1)!a_{-1} + g^{(N-1)}(z). \tag{26.6}
$$

Pass to the limit in (26.6) as $z \to z_0$, one has

$$
\lim_{z\to z_0}\frac{d^{N-1}}{dz^{N-1}}(z-z_0)^N f(z) = (N-1)!a_{-1} + g^{(N-1)}(z_0).
$$

But $g^{(N-1)}(z_0) = 0$ (make sure you see it) and we arrive at (26.3). □

Remark 26.4. *Theorem 26.3 gives a remarkably useful way to evaluate contour integrals without doing any integration. Indeed equations* (26.2) *and* (26.3) *don't have any integration but series expansion and/or differentiation.*

Corollary 26.5. *If z_0 is a removable singularity of f then* $\operatorname{Res}(f, z_0) = 0$.

Indeed, look at equation (26.2).

Remark 26.6. *One can of course think of a point of analyticity z_0 of a function f as a removable singularity. Then* $\operatorname{Res}(f, z_0)$ *will be zero.*

Corollary 26.7. *If z_0 is a simple pole, i.e. $N = 1$, then*

$$
\operatorname{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z). \tag{26.7}
$$

Indeed, take in (26.3) $N = 1$. Recalling that $0! = 1$ and the zero order derivative is the function itself.

Corollary 26.8. *If z_0 is a simple pole of f and f can be represented in some neighborhood of z_0 as*

$$
f(z) = \frac{\varphi(z)}{\psi(z)}
$$

with some analytic functions φ, ψ such that $\varphi(z_0) \neq 0$, $\psi(z_0) = 0$ then

$$
\operatorname{Res}(f, z_0) = \frac{\varphi(z_0)}{\psi'(z_0)}. \tag{26.8}
$$

The proof is left as an exercise.

The following examples should demonstrate how useful Theorem 26.3 and its Corollaries are.

EXAMPLE 26.9. *Evaluate* $\int_{C_1(0)} e^{1/z}dz$. *By Theorem 26.3*

$$\int_{C_1(0)} e^{1/z}dz = 2\pi i \operatorname{Res}(e^{1/z}, 0) \overset{(26.2)}{=} 2\pi i a_{-1}.$$

But we know the Laurent expansion for $e^{1/z}$

$$e^{1/z} = \sum_{n\geq 0} \frac{1}{n!}(z)^{-n} = 1 + \frac{1}{z} + \cdots \quad \Rightarrow \quad a_{-1} = 1.$$

So,

$$\int_{C_1(0)} e^{1/z}dz = 2\pi i.$$

EXAMPLE 26.10. *Evaluate* $\int_{C_1(0)} \frac{\sin z}{z^3}dz$. *From Example 24.3(2), we know* $\frac{\sin z}{z^3}$ *has a pole of order 2 at* $z = 0$. *Theorem 26.3 gives*

$$\int_{C_1(0)} \frac{\sin z}{z^3}dz \overset{(26.3)}{=} 2\pi i \lim_{z\to 0} \frac{d}{dz}\frac{\sin z}{z} = 2\pi i \lim_{z\to 0} \frac{z\cos z - \sin z}{z^2}$$

$$\overset{\text{l'Hospital}}{=} 2\pi i \lim_{z\to 0} \frac{-z\sin z + \cos z - \cos z}{2z} = 0.$$

Note that we could have obtained this result using (26.2).

EXAMPLE 26.11. *Evaluate* $\int_{C_1(0)} \frac{\sin z}{z^2}dz$. *We note that*

$$\lim_{z\to 0} z\frac{\sin z}{z^2} = 1 \neq 0. \tag{26.9}$$

Hence, by Proposition 24.4, $z = 0$ *is a simple pole for* $\frac{\sin z}{z^2}$. *Therefore we have*

$$\int_{C_1(0)} \frac{\sin z}{z^2}dz = 2\pi i \operatorname{Res}\left(\frac{\sin z}{z^2}, 0\right) \overset{(26.7)}{=} 2\pi i \lim_{z\to 0} z\frac{\sin z}{z^2} \overset{(26.9)}{=} 2\pi i.$$

EXAMPLE 26.12. *Evaluate* $\int_{C_1(i)} \frac{\sin z}{z^2+1}dz$. *Let* $\varphi(z) = \sin z$, $\psi(z) = z^2 + 1$. *Note that* φ *is analytic on* $\mathbb{D}_1(i)$, *and since*

$$\lim_{z\to i}(z-i)\frac{\sin z}{z^2+1} = \frac{\sin i}{2i} \neq 0,$$

it follows by Proposition 24.4 that $z = i$ *is a simple pole of* $\frac{\sin z}{z^2+1}$. *Hence*

$$\int_{C_1(i)} \frac{\sin z}{z^2+1}dz = 2\pi i \operatorname{Res}\left(\frac{\varphi(z)}{\psi(z)}, i\right) \overset{(26.8)}{=} 2\pi i \frac{\varphi(i)}{\psi'(i)} = 2\pi i\frac{\sin i}{2i} = \frac{\pi i}{2}\left(e - \frac{1}{e}\right).$$

2. Residue at ∞

DEFINITION 26.13. *Let $f(z)$ be analytic outside of a bounded region Ω. Then for any simply connected closed contour C enclosing Ω (see Figure 1)*

$$\operatorname{Res}(f,\infty) \overset{def}{=} \frac{1}{2\pi i}\int_{-C} f(z)dz.$$

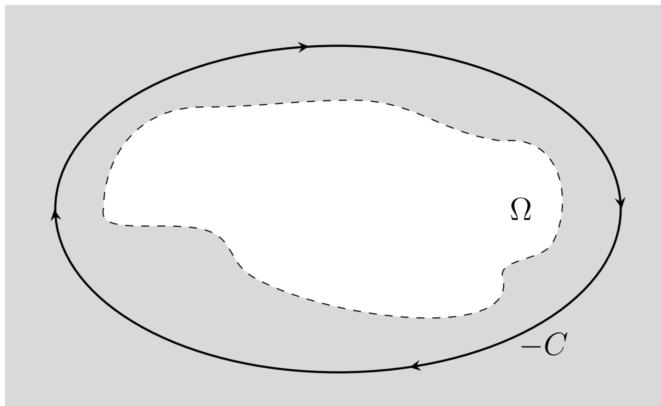


FIGURE 1

THEOREM 26.14. *Let f be analytic outside of a bounded region Ω. Then*

(1)
$$\operatorname{Res}(f,\infty) = -a_{-1}$$
where a_{-1} is the corresponding Laurent coefficient in the Laurent expansion on $|z| > R$ with R so large that $\mathbb{D}_R(0) \supset \Omega$.

(2)
$$\operatorname{Res}(f,\infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right).$$

The proof is left as an exercise.

REMARK 26.15. *Corollary 26.5 stated that if z_0 is a removable singularity of f, then $\operatorname{Res}(f, z_0) = 0$. This is only valid for z_0 finite. I.e. if $z = \infty$ is a removable singularity of f, $\operatorname{Res}(f,\infty)$ need not be zero. This is the main difference between isolated removable singularities and removable singularities at ∞.*

Exercises

Exercise 26.1 Prove Corollary 26.8.

Exercise 26.2 Prove Theorem 26.14.

Exercise 26.3 Give a counterexample supporting the claim in Remark 26.15.

Exercise 26.4 Do Example 16.3(a) using Theorem 26.14.

LECTURE 27

The Residue Theorem

We now finally get to the main statement of this course - the residue theorem which is originally due to, guess whom, Cauchy.

Theorem 27.1 (The Residue Theorem). *Let f be analytic on a simply connected region E except at a finite number of isolated singularities $z_1, z_2, ..., z_n$. Then for any simply connected closed contour $C \subset E$ enclosing $\{z_k\}_{k=1}^n$ (see Figure 1)*

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k). \tag{27.1}$$

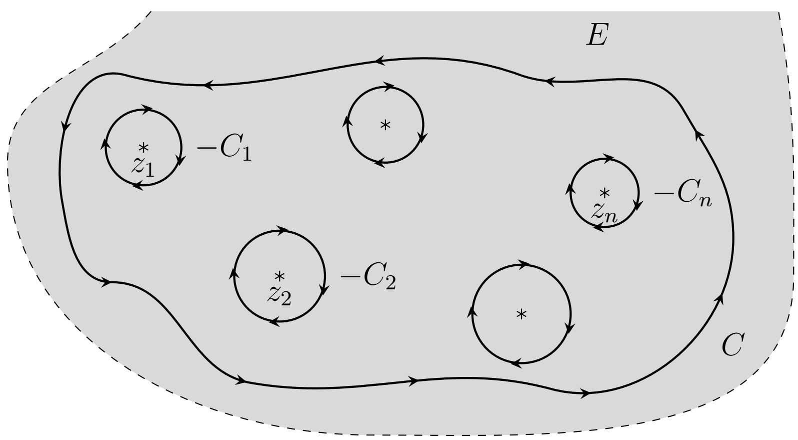


Figure 1

Proof. Consider the contour (see Figure 1)

$$\Gamma = C \bigcup \left\{ \bigcup_{k=1}^{n} (-C_k) \right\}$$

where $C_k = C_\rho(z_k)$ with some ρ small enough. By the Cauchy Theorem for multiply connected regions (Theorem 16.1)

$$0 = \int_\Gamma f(z)dz = \int_C f(z)dz + \sum_{k=1}^{n} \underbrace{\int_{-C_k} f(z)dz}_{=-2\pi i \operatorname{Res}(f,z_k)}$$

and (27.1) follows. □

REMARK 27.2. *Coupled with the efficient ways to evaluate residues, the Residue Theorem becomes a particularly powerful tool in evaluating integrals (not necessarily contour). It will take us a few lectures to learn how to use this tool.*

If in Theorem 27.1 $E = \mathbb{C}$ then we also have

THEOREM 27.3. *If f is analytic on $\mathbb{C} \setminus \{z_k\}_{k=1}^n$ then*

$$\sum_{k=1}^{n} \operatorname{Res}(f, z_k) + \operatorname{Res}(f, \infty) = 0. \tag{27.2}$$

The proof is left as an exercise.

Theorem 27.3 allows us to generalize the Cauchy formula to the case of certain unbounded regions.

THEOREM 27.4. *Let Ω be a bounded region and $E = \mathbb{C} \setminus \overline{\Omega}$. Assume f is analytic on E and ∞ is its removable singularity. Then for any simple contour C in E enclosing the region Ω*

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = \begin{cases} f(\infty), & z_0 \in \operatorname{Int} C \\ f(\infty) - f(z_0), & z_0 \in \operatorname{Ext} C \end{cases}$$

where $f(\infty) = \lim_{z \to \infty} f(z)$.

The proof is left as an exercise.

The following examples show some applications of Theorem 27.3.

EXAMPLE 27.5. *Evaluate $\int_C \frac{dz}{(z+1)(z^2+1)}$ where C is as in Figure 2 (it was in Example 16.3(b)).*

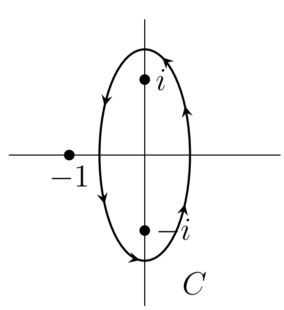


FIGURE 2

Solution We do it in two ways

(1) By formula (27.1): $\dfrac{1}{(z+1)(z^2+1)}$ is analytic in $\operatorname{Int} C \setminus \{\pm i\}$. Hence

$$\begin{aligned}
\int_C f(z)dz &\overset{(27.1)}{=} 2\pi i \left\{\operatorname{Res}(f,i) + \operatorname{Res}(f,-i)\right\} \\
&= 2\pi i \left\{ \operatorname*{Res}_{z=i} \underbrace{\frac{1}{z^2+1}}_{\frac{1}{\psi}} \underbrace{\frac{1}{z+1}}_{\varphi} + \operatorname*{Res}_{z=-i} \frac{1}{z^2+1}\frac{1}{z+1} \right\} \\
&\overset{(26.8)}{=} 2\pi i \left\{ \frac{1}{2z}\frac{1}{z+1}\bigg|_{z=i} + \frac{1}{2z}\frac{1}{z+1}\bigg|_{z=-i} \right\} \\
&= 2\pi i \left(\frac{1}{2(i)(i+1)} + \frac{1}{2(-i)(-i+1)} \right) \\
&= \pi \left(\frac{1}{1+i} - \frac{1}{1-i} \right) = -\pi i.
\end{aligned}$$

(2) By formula (27.2):

$$\begin{aligned}
&\int_C f(z)dz + \int_{C_1(-1)} f(z)dz + 2\pi i \operatorname{Res}(f,\infty) = 0 \\
\Rightarrow &\int_C f(z)dz = -2\pi i \{\operatorname{Res}(f,-1) + \underbrace{\operatorname{Res}(f,\infty)}_{=0 \text{ by Exercise 26.4}} \} \\
&\qquad = -2\pi i (\frac{1}{2} + 0) = -\pi i.
\end{aligned}$$

Example 27.6. *Evaluate* $\displaystyle\int_{C_4(0)} \frac{dz}{\sin z}$.

Solution The function $\dfrac{1}{\sin z}$ has three simple poles in $\operatorname{Int} C_4(0)$: $z_1 = -\pi, z_2 = 0, z_3 = \pi$.

$$\begin{aligned}
\int_{C_4(0)} \frac{dz}{\sin z} &= 2\pi i \{\operatorname{Res}(f,-\pi) + \operatorname{Res}(f,0) + \operatorname{Res}(f,\pi)\} \\
&= 2\pi i \left\{ \frac{1}{\cos z}\bigg|_{-\pi} + \frac{1}{\cos z}\bigg|_{0} + \frac{1}{\cos z}\bigg|_{\pi} \right\} = -2\pi i.
\end{aligned}$$

Exercises

Exercise 27.1 Prove Theorem 27.3.

Exercise 27.2 Evaluate $\displaystyle\int_{C_2(0)} \frac{dz}{1+z^4}$ in two different ways by (27.1) and (27.2).

Exercise 27.3

(1) Prove that if f is even then $\displaystyle\int_{C_R(0)} f(z)dz = 0$ for any $R > 0$ such that no singularities are on the contour.
(2) Conclude then that $\mathrm{Res}(f, 0) = 0$.
(3) Use this fact to do Exercise 27.2 in yet another way.

Exercise 27.4 Prove Theorem 27.4.

LECTURE 28

Applications of the Residue Theorem to Evaluate Some Definite Integrals

In this and a few of the following lectures, we are going to show that the residue theorem turns out to be an indispensable tool in computing some definite integrals which can hardly be computed by any other means.

1. Integrals of the type $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$

Let us consider the following type of integrals

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$$

where R is a rational function. Such integrals are called trigonometric and R is typically a real function. Assume that $R(\cos\theta, \sin\theta)$ is continuous for $\theta \in [0, 2\pi]$. Since $\theta \in [0, 2\pi]$, $z := e^{i\theta} \in C_1(0)$ i.e., z runs along the unit circle when θ changes from 0 to 2π (as seen in Figure 1).

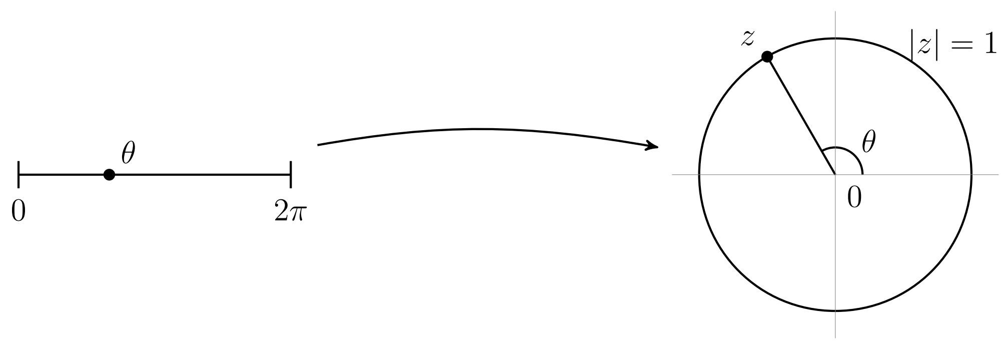


Figure 1

We have

$$dz = izd\theta \Rightarrow d\theta = \frac{dz}{iz},$$
$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right),$$
$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

So, making these substitutions we get

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta = \int_{|z|=1} \tilde{R}(z) dz, \tag{28.1}$$

where $\tilde{R}(z)$ is a new rational function of z such that

$$\tilde{R}(z) = \frac{1}{iz} R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) = \frac{P_n(z)}{Q_m(z)}$$

for some polynomials $P_n(z)$, $Q_m(z)$. The function $\tilde{R}(z)$ is analytic inside $C_1(0)$ except for a finite number of poles $\{z_1, z_2, ..., z_N\}$, $N \le m$. By the Residue Theorem,

$$I = 2\pi i \sum_{k=1}^{N} \operatorname{Res}(\tilde{R}(z), z_k). \tag{28.2}$$

Remark 28.1. *If the function R in* (28.1) *is real, then so is I. However, the RHS of* (28.2) *looks complex. Make sure that in your computations the RHS will eventually turn into real!*

Example 28.2. *Compute*

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \qquad a \in \mathbb{R}, |a| < 1. \tag{28.3}$$

Solution Put $z = e^{i\theta}$. We have

$$I = \int_{|z|=1} \frac{dz}{iz\left(1 + a\frac{z+\frac{1}{z}}{2}\right)} = \frac{2}{i}\int_{|z|=1} \underbrace{\frac{1}{az^2+2z+a}}_{R(z)} dz.$$

The function $R(z)$ has two simple poles z_1, z_2 which are the solutions to $az^2+2z+a=0$. One has

$$z_{1,2} = \frac{-1 \pm \sqrt{1-a^2}}{a}.$$

Note that

$$|z_2| = \frac{1+\sqrt{1-a^2}}{|a|} = \frac{1}{|a|} + \sqrt{\frac{1}{a^2} - 1} > \frac{1}{|a|} > 1.$$

But, by the Vieta Theorem $z_1 z_2 = 1$ and hence $|z_1| = \frac{1}{|z_2|} < 1$. Thus only $z_1 \in \operatorname{Int} C_1(0)$ and hence, by the Residue Theorem

$$\operatorname{Res}(R, z_1) = \operatorname{Res}\left(\frac{1}{az^2+2z+a}, z_1\right) = \frac{1}{2az_1+2} = \frac{2}{2(-1+\sqrt{1-a^2}+1)} = \frac{1}{2\sqrt{1-a^2}}.$$

Therefore

$$I = 2\pi i \frac{2}{i} \operatorname{Res}(R, z_1) = \frac{2\pi}{\sqrt{1-a^2}}.$$

If you remember, in calculus, evaluation of integrals like (28.3) would take much more time and space.

2. Integrals of the type $\int_{-\infty}^{\infty} f(x)dx$

The computation of such integrals is based upon

LEMMA 28.3. *Let f be analytic on $\mathbb{C}^+ \setminus \{z_k\}_{k=1}^n$. If there exists $\rho > 0$, $M > 0$, $\delta > 0$ such that*

$$|f(z)| \le \frac{M}{|z|^{1+\delta}} \qquad \forall z \in \mathbb{C}^+, |z| > \rho \tag{28.4}$$

then

$$\lim_{R\to\infty} \int_{C_R^+} f(z)dz = 0,$$

where $C_R^+ := C_R(0) \cap \mathbb{C}^+$ as in Figure 2.

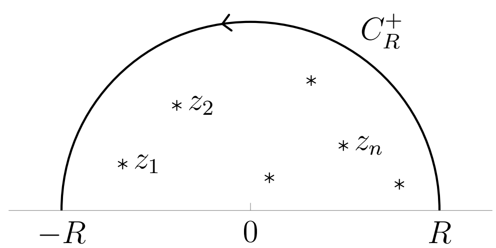


FIGURE 2

PROOF. Let R be big enough so that all $\{z_k\}_{k=1}^n$ are inside $C_R(0)$. Then

$$\left|\int_{\mathbb{C}_R^+} f(z)dz\right| \le \int_{\mathbb{C}_R^+} |f(z)||dz| \overset{(28.4)}{\le} \int_{\mathbb{C}_R^+} \frac{M}{|z|^{1+\delta}}|dz|$$
$$= \frac{M}{R^{1+\delta}} \int_{\mathbb{C}_R^+} |dz| = \frac{M}{R^{1+\delta}}\pi R = \frac{\pi M}{R^\delta} \to 0 \text{ , as } R \to \infty.$$

□

EXAMPLE 28.4. $f(z) = \dfrac{1}{1+z^4}$ *satisfies the conditions of Lemma 28.3. Indeed, $f(z)$ has two isolated singularities in $\mathbb{C}^+$ and*

$$|f(z)| = \frac{1}{|1+z^4|} \overset{\text{triangle ineq}}{\le} \frac{1}{|z|^4 - 1} = \frac{1}{1-|z|^{-4}}\frac{1}{|z|^4}.$$

Fix $\rho = 2$ (e.g.) and consider

$$M := \sup_{|z|\ge 2} \frac{1}{1-|z|^{-4}} = (\inf_{|z|\ge 2}(1-|z|^{-4}))^{-1}$$
$$= (1 - \sup_{|z|\ge 2}|z|^{-4})^{-1} = (1-2^{-4})^{-1} = \frac{16}{15} > 0.$$

Thus

$$|f(z)| \le \sup_{|z|\ge 2} \frac{1}{1-|z|^{-4}}\frac{1}{|z|^4} = \frac{M}{|z|^4}, \quad \forall |z| > 2.$$

DEFINITION 28.5. *Let $f(x) : \mathbb{R} \to \mathbb{C}$. A function $f(z) : \mathbb{C}^+ \to \mathbb{C}$ is called an analytic continuation of $f(x)$ into $\mathbb{C}^+$ if $f(z)$ is analytic on $\mathbb{C}^+$ except for a countable number of isolated singularities and $f(z)|_{z=x} = f(x)\ \forall x \in \mathbb{R}$.*

EXAMPLE 28.6. *Let $f(x) = \dfrac{1}{1+x^4}$. Consider $f(z) = \dfrac{1}{1+z^4}$. Notice $f(z)$ has two poles in $\mathbb{C}^+$ and is analytic everywhere else on $\mathbb{C}^+$. Moreover*

$$\left.\frac{1}{1+z^4}\right|_{z=x} = \frac{1}{1+x^4}.$$

Thus $\dfrac{1}{1+z^4}$ is indeed an analytic continuation of $\dfrac{1}{1+x^4}$ into $\mathbb{C}^+$.

Not every $f : \mathbb{R} \to \mathbb{C}$ has an analytic continuation into $\mathbb{C}^+$ (we present this fact without proof).

THEOREM 28.7. *Let $f(x)$ have an analytic continuation into $\mathbb{C}^+$ and its analytic continuation $f(z)$ satisfies the conditions of Lemma 28.3 and has no poles on $\mathbb{R}$. Then*

$$\int_{\mathbb{R}} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k) \tag{28.5}$$

where $\{z_k\}_{k=1}^n$ are the poles of f in $\mathbb{C}^+$.

PROOF. By condition $f(z)$ is analytic on $\mathbb{C}^+ \setminus \{z_k\}_{k=1}^N$ and by the Residue Theorem

$$\int_{\Gamma_R} f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}\{f(z), z_k\} \tag{28.6}$$

where Γ_R is as in Figure 3 and R is large enough so that all the poles of $f(z)$ get inside of Γ_R.

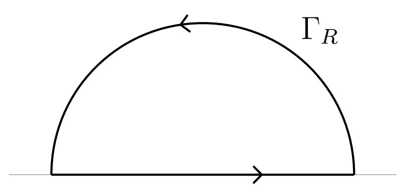


FIGURE 3

But $\displaystyle\int_{\Gamma_R} = \int_{-R}^{R} + \int_{C_R^+}$, where C_R^+ as in Lemma 28.3. It follows then from (28.6) that

$$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{N} \operatorname{Res}\{f(z), z_k\} - \int_{C_R^+} f(z)dz$$

Pass now to the limit as $R \to \infty$ in this equation and we get

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{N} \operatorname{Res}\{f(z), z_k\} - \underbrace{\lim_{R\to\infty} \int_{C_R^+} f(z)dz}_{=0 \text{ by Lemma 28.3}}.$$

and the theorem is proven. $\square$

EXAMPLE 28.8. *Prove that* $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{2}$.

By Example 28.6,

$$f(x) = \frac{1}{1+x^4}$$

has an analytic continuation into $\mathbb{C}^+$*, with two poles in* $\mathbb{C}^+$*:*

$$z_1 = \frac{\sqrt{2}}{2}(1+i) = e^{i\pi/4},$$
$$z_2 = \frac{\sqrt{2}}{2}(-1+i) = e^{i3\pi/4}.$$

By Example 28.4, $f(z)$ *satisfies the conditions in Lemma 28.3 and hence*

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} &\overset{\text{Thm 28.7}}{=} 2\pi i \left(\operatorname*{Res}_{z=z_1} f + \operatorname*{Res}_{z=z_2} f \right) \\ &= 2\pi i \left(\frac{1}{4z_1^3} + \frac{1}{4z_2^3} \right) = \frac{\pi i}{2}\left(e^{-i3\pi/4} + e^{-i\pi/4} \right) \\ &= \frac{\pi i}{2}\frac{\sqrt{2}}{2}\left(-1-i+1-i\right) = \frac{\pi\sqrt{2}}{2}.\end{aligned}$$

REMARK 28.9. $\mathbb{C}^+$ *is chosen for convenience. All arguments come through for* $\mathbb{C}^-$ *too.*

Exercises

Exercise 28.1 Evaluate by residue $\displaystyle\int_0^{\pi} \frac{d\theta}{1+\sin^2\theta}$.

Exercise 28.2 Show $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos^2\theta} = \frac{2\pi}{\sqrt{a}\sqrt{a+b}}$ for $0<b<a$.

Exercise 28.3 Show $\displaystyle\int_0^{\infty} \frac{dx}{(x^2+1)^3} = \frac{3\pi}{16}$.

Exercise 28.4 Show $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$.

Exercise 28.5 Show that for any rational function $R(z) = \dfrac{p(z)}{q(z)}$ where $\deg q - \deg p \geq 2$, $R(z)$ is subject to the conditions of Lemma 28.3.

LECTURE 29

Applications of the Residue Theorem; Some Integrals of the Type $\int_0^\infty f(x)dx$

Consider now the next type of improper integrals which can be done by residues

$$I = \int_0^\infty \frac{dx}{x^\alpha + 1} \quad , \quad \alpha > 1. \tag{29.1}$$

Use the substitution $x = e^t$, $t \in \mathbb{R}$. Then

$$I = \int_{-\infty}^\infty \frac{e^t dt}{e^{\alpha t} + 1}. \tag{29.2}$$

The function does admit an analytic continuation into $\mathbb{C}^+$, and it has infinitely many simple poles z_k subject to

$$e^{\alpha z_k} + 1 = 0 \quad \Leftrightarrow \quad z_k = \frac{i\pi}{\alpha}(2k+1) = i(2k+1)a, \quad a = \frac{\pi}{\alpha}, k \in \mathbb{Z}. \tag{29.3}$$

All the poles z_k are on the imaginary axis and equally spaced. The contour Γ_R we used to evaluate $\displaystyle\int_{-\infty}^\infty f(x)dx$ is no longer good and we have to make up something different.

Choosing a right contour is <u>crucial</u> and the most <u>difficult part</u> of the Residue Theorem approach to certain definite integrals. Nothing <u>guarantees</u> that this will work at all.

Consider the contour Γ_R as on Figure 1.

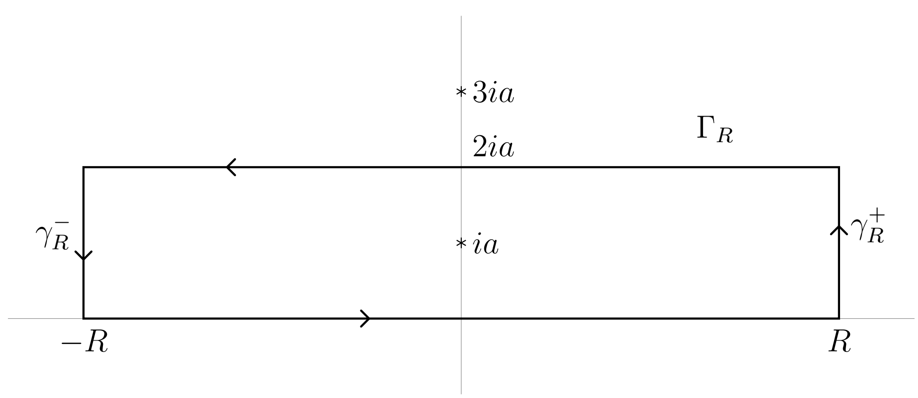


FIGURE 1

Let $f(z) = \dfrac{e^z}{e^{\alpha z}+1}$ be the analytic continuation of $f(t)$ into $\mathbb{C}^+$. By the Residue Theorem

$$\int_{\Gamma_R} f(z)dz = 2\pi i \operatorname{Res}(f, ia) = 2\pi i \left.\frac{e^z}{\alpha e^{\alpha z}}\right|_{z=ia} = 2\pi i \frac{e^{ia}}{-\alpha} = -2iae^{ia}. \tag{29.4}$$

On the other hand

$$\int_{\Gamma_R} = \underbrace{\int_{-R}^{R}}_{I_1} + \underbrace{\int_{R+2ia}^{-R+2ia}}_{I_2} + \underbrace{\int_{\gamma_R^+}}_{I_3} + \underbrace{\int_{\gamma_R^-}}_{I_4}. \tag{29.5}$$

Let's take care of each integral I_k separately.

$$I_1 = \int_{-R}^{R} \frac{e^t}{e^{\alpha t}+1} dt \quad \underset{R\to\infty}{\longrightarrow} \quad \int_{\mathbb{R}} \frac{e^t}{e^{\alpha t}+1} dt = I, \tag{29.6}$$

where I, again, is our original integral. For I_2,

$$\begin{aligned} I_2 &= \int_{R+2ia}^{-R+2ia} \frac{e^z}{e^{\alpha z}+1} dz = \int_{R}^{-R} \frac{e^{t+2ia}}{e^{\alpha(t+2ia)}+1} dt \\ &= -e^{2ia} \int_{-R}^{R} \frac{e^t}{e^{\alpha t}\underbrace{e^{2i\pi}}_{=1}+1} dt = -e^{2ia} I_1. \end{aligned} \tag{29.7}$$

As for $I_{3,4}$, we cannot evaluate them explicitly but we show that they tend to 0 as $R \to \infty$.

$$\begin{aligned} I_{3,4} &= \int_{\gamma_R^\pm} f(z)dz = \pm \int_{\pm R}^{\pm R+2ia} \frac{e^z}{e^{\alpha z}+1} dz = \pm \int_0^{2a} \frac{e^{\pm R+iy}}{e^{\alpha(\pm R+iy)}+1} i dy \\ \Rightarrow |I_{3,4}| &\overset{\text{triangle ineq}}{\leq} \int_0^{2a} \frac{e^{\pm R}}{|e^{\pm\alpha R} e^{i\alpha y}+1|} dy \overset{\text{triangle ineq}}{\leq} \int_0^{2a} \frac{e^{\pm R}}{|e^{\pm \alpha R}-1|} dy \\ &= \frac{2a}{|e^{\pm(\alpha-1)R} - e^{\mp R}|} \longrightarrow 0, \quad R \to \infty. \end{aligned} \tag{29.8}$$

Combining (29.4), (29.5), (29.7) we have

$$-2iae^{ia} = \left(1 - e^{2ia}\right) I_1 + I_3 + I_4.$$

But as $R \to \infty$, $I_1 \to I$ by (29.6), and $I_{3,4} \to 0$ by (29.8). Hence

$$-2iae^{ia} = \left(1 - e^{2ia}\right) I$$

and

$$I = \frac{2iae^{ia}}{e^{2ia}-1} = \frac{2i}{e^{ia} - e^{-ia}} a = \frac{a}{\sin a}.$$

Recalling that $a = \dfrac{\pi}{\alpha}$ we get

$$\int_0^\infty \frac{dx}{x^\alpha + 1} = \frac{\pi}{\alpha \sin \frac{\pi}{\alpha}}. \tag{29.9}$$

Done!

Exercises

Exercise 29.1 If $n, m \in \mathbb{N}$ and $n \leq 2m - 2$, show that
$$\int_{\mathbb{R}} \frac{x^n dx}{1 + x^{2m}} = \frac{(1 + (-1)^n)\,\pi}{2m \sin \dfrac{\pi}{2m}(n+1)}.$$

Exercise 29.2 The Fourier transform of a function f such that $\displaystyle\int |f(x)|^2 \, dx$ converges is defined by the following:
$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\lambda x} f(x) dx.$$
Evaluate the Fourier transform of the function $f(x) = \alpha^2 \operatorname{sech}^2(\alpha x)$ where $\alpha \neq 0$.

Exercise 29.3 Prove (29.9) for $\alpha = 3$ using the contour in Figure 2.

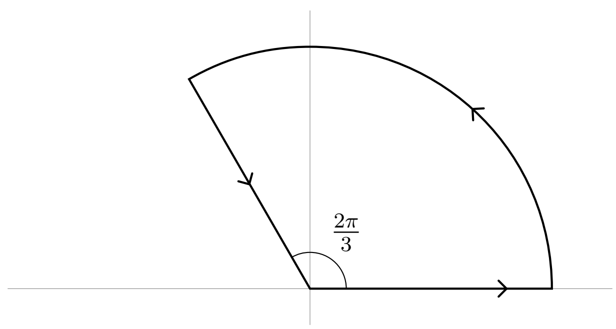


FIGURE 2

Exercise 29.4 Show that if $0 < \alpha < 1$ then
$$\int_0^\infty \frac{x^\alpha}{(x+1)^2} dx = \frac{\pi\alpha}{\sin \pi\alpha}.$$

LECTURE 30

The Art of Contour Integration: Integrals of the Type $\int_{-\infty}^{\infty} e^{iax} f(x)dx$ and Jordan's Lemma

Integrals of the form $\displaystyle\int_{-\infty}^{\infty} e^{iax} f(x)dx$ are particularly common in physics and engineering due to their connection to Fourier Analysis which is at the core of applied analysis. Evaluation of such integrals is based upon the following famous lemma.

LEMMA 30.1 (Jordan's Lemma). *Let $f(z)$ be analytic on $\mathbb{C}^+ \setminus \{z_k\}_{k=1}^n$ and suppose*

$$\lim_{R\to\infty} \sup_{0\le\theta\le\pi} |f(Re^{i\theta})| = 0. \tag{30.1}$$

If $a > 0$ then

$$\lim_{R\to\infty} \int_{C_R^+} e^{iaz} f(z)dz = 0 \tag{30.2}$$

where $C_R^+ = C_R(0) \cap \mathbb{C}^+$.

PROOF. Set $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$. Then

$$\int_{C_R^+} e^{iaz} f(z)dz = iR \int_0^\pi e^{iaR\cos\theta - aR\sin\theta} f(Re^{i\theta}) e^{i\theta} d\theta.$$

By the triangle inequality

$$\begin{aligned}\left|\int_{C_R^+} e^{iaz} f(z)dz\right| &\le R \int_0^\pi e^{-aR\sin\theta} |f(Re^{i\theta})| d\theta \\ &\le \sup_{0\le\theta\le\pi} |f(Re^{i\theta})| \underbrace{R \int_0^\pi e^{-aRsin\theta} d\theta}_{=I_R}. \qquad (30.3)\end{aligned}$$

Consider I_R. We can see in Figure 1 that $\sin\theta \ge \dfrac{2}{\pi}\theta$ for all $0 \le \theta \le \pi/2$. We also have

$$\int_0^\pi e^{-aR\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-aR\sin\theta} d\theta.$$

Therefore

$$\begin{aligned} I_R &= 2R\int_0^{\pi/2} e^{-aR\sin\theta} d\theta \le 2R \int_0^{\pi/2} e^{-\frac{2}{\pi}aR\theta} d\theta \\ &= 2R \frac{1}{-\frac{2}{\pi}aR} e^{-\frac{2}{\pi}aR\theta}\Big|_0^{\pi/2} = \frac{\pi}{a}(1 - e^{-aR}) \le \frac{\pi}{a}. \qquad (30.4)\end{aligned}$$

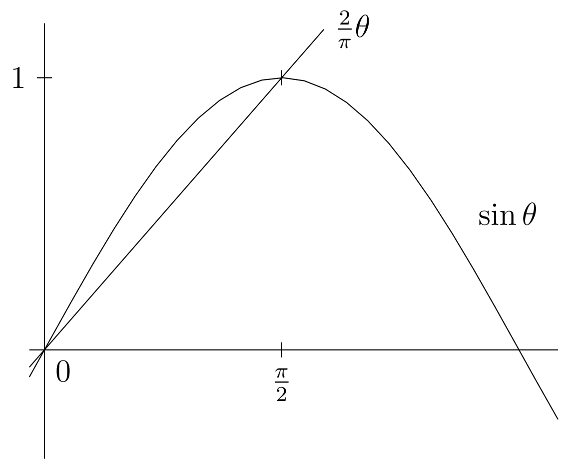


FIGURE 1

Combining (30.3) and (30.4) we get

$$\left|\int_{C_R^+} e^{iaz} f(z)dz\right| \le \frac{\pi}{a} \underbrace{\sup_{0\le\theta\le\pi} |f(Re^{i\theta})|}_{\to 0,\ R\to\infty \text{ by } (30.2)} . \qquad \square$$

Notice that Jordan's Lemma looks very similar to Lemma 28.3. The difference is the rate of decay of $f(z)$ as $z \to \infty$ for $z \in \mathbb{C}^+$. Recall that we require:
in Lemma 28.3

$$|R| \sup_{0\le\theta\le\pi} |f\left(Re^{i\theta}\right)| \to 0,\ R \to \infty,$$

and in Lemma 30.1

$$\sup_{0\le\theta\le\pi} |f(Re^{i\theta})| \to 0,\ R \to \infty.$$

The reason for the different requirements on the rate of decay is the additional factor e^{iaz} in (30.2). Notice that if $a > 0$ and $z \to \infty$ along any ray $Re^{i\theta}$ in $\mathbb{C}^+$ different from $\mathbb{R}_\pm$, then $e^{iaz} = e^{-aR\sin\theta + iaR\cos\theta}$ decays rapidly. If $z \to \pm\infty$ along $\mathbb{R}_\pm$ respectively, then e^{iaz} shows oscillatory behaviour. In other words, it is the decay of e^{iaz} in $\mathbb{C}^+$ and oscillations in $\mathbb{R}$ that ensure (30.2) happens while $\int_{C_R^+} |f(z)|dz$ may blow up. We now apply Jordan's Lemma to evaluate a broad class of the so-called Fourier integrals.

THEOREM 30.2. *Suppose $f(x)$ is continuous on $\mathbb{R}$ and admits an analytic continuation $f(z)$ into $\mathbb{C}^+$. If $f(z)$ is subject to the conditions in Jordan's Lemma, then the improper integral ($a > 0$)*

$$\lim_{R\to\infty} \int_{-R}^{R} e^{iax} f(x)dx = \int_{\mathbb{R}} e^{iax} f(x)dx$$

converges and

$$\int_{\mathbb{R}} e^{iax} f(x)dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\{e^{iaz} f(z), z_k\}$$

where $\{z_k\}_{k=1}^n$ *are the poles of* $f(z)$ *in* $\mathbb{C}^+$.

The proof is left as an exercise.

REMARK 30.3. *The assumption of continuity of* $f(x)$ *can be relaxed, but it is not our objective to deal with this in this lecture. One particularly important case where* $f(x)$ *is discontinuous will be covered in the next lecture.*

REMARK 30.4. *If* $a < 0$ *then one can derive a similar theorem using* f *in* $\mathbb{C}^-$ *or using complex conjugates. The proof is left as an exercise.*

EXAMPLE 30.5. *Evaluate*

$$I := \int_{\mathbb{R}} \frac{\cos ax}{x^2 + b^2} dx \ ; \quad a, b > 0.$$

<u>Solution</u> Since $\cos ax = \mathrm{Re}\, e^{iax}$

$$I = \int_{\mathbb{R}} \frac{\mathrm{Re}\, e^{iax}}{x^2 + b^2} dx = \mathrm{Re} \int_{\mathbb{R}} \frac{e^{iax}}{x^2 + b^2} dx.$$

Observe that the function $f(x) = \dfrac{1}{x^2 + b^2}$ is continuous and its analytic continuation $f(z) = \dfrac{1}{z^2 + b^2}$ satisfies the conditions of Jordan's Lemma. By Theorem 30.2 we have

$$I = \mathrm{Re}\left(2\pi i \,\mathrm{Res}\left(\frac{e^{iaz}}{z^2 + b^2}, ib\right)\right) = \mathrm{Re}\left(2\pi i \frac{e^{-ab}}{2ib}\right) = \frac{\pi}{b} e^{-ab}.$$

Thus,

$$I = \int_{\mathbb{R}} \frac{\cos ax}{x^2 + b^2} dx = \frac{\pi}{b} e^{-ab}.$$

Exercises

Exercise 30.1 Prove Theorem 30.2.

Exercise 30.2 Show the following

(1) Let $a < 0$. If $f(z)$ is analytic in $\mathbb{C}^- \setminus \{z_k\}_{k=1}^n$ and

$$\lim_{R \to \infty} \sup_{-\pi \le \theta \le 0} \left| f(Re^{i\theta}) \right| = 0$$

then

$$\lim_{R \to \infty} \int_{C_R^-} e^{iaz} f(z) dz = 0$$

where $C_R^- = C_R(0) \cap \mathbb{C}^-$.

(2) If f satisfies the conditions above then for any $a < 0$

$$\int_{\mathbb{R}} e^{iax} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left\{e^{iaz} f(z), z_k\right\}$$

where $\{z_k\}_{k=1}^{N}$ are the poles in $\mathbb{C}^-$.

(3) Alternately use Theorem 30.2 to derive another formula for the case $a < 0$.

Exercise 30.3 Evaluate

$$\int_{\mathbb{R}} \frac{x \sin ax}{x^2 + b^2} dx.$$

Exercise 30.4 Show that

$$\int_{\mathbb{R}} \frac{\cos x}{(x^2+1)^2} dx = \frac{\pi}{e}$$

and use this result to derive a formula for

$$\int_{\mathbb{R}} \frac{\cos \frac{x}{a}}{(x^2+a^2)^2} dx \ , \quad a > 0.$$

LECTURE 31

The Art of Contour Integration: Fresnel Integrals, Indented Contours

1. Fresnel Integrals

In this subsection we are concerned with computing two more integrals ($a \in \mathbb{R}$):

$$F_1(a) = \int_0^\infty \cos(ax^2)dx, \tag{31.1}$$

$$F_2(a) = \int_0^\infty \sin(ax^2)dx. \tag{31.2}$$

These integrals were introduced by the French mathematician and physicist Fresnel in the connection with diffraction theory. Integrals (31.1) and (31.2) are now referred to as the Fresnel integrals.

Without loss of generality we may assume $a > 0$ and then, substituting $\sqrt{a}x \to x$ in (31.1) and (31.2), we have

$$F_1(a) = \frac{1}{\sqrt{a}} \int_0^\infty \cos x^2 dx, \tag{31.3}$$

$$F_2(a) = \frac{1}{\sqrt{a}} \int_0^\infty \sin x^2 dx. \tag{31.4}$$

Note that the integrals on the right-hand side of (31.3) and (31.4) no longer depend on a.

The question on everyone's mind should be why integrals (31.1) and (31.2) converge. The sin and cos functions don't decay at infinity! But the Fresnel integrals do converge. It happens due to the rapid oscillation of $\cos x^2$ and $\sin x^2$ as $x \to \infty$, but convergence is very difficult (if not impossible) to prove without contour integration.

We now introduce the contour integral

$$I := \int_C e^{iz^2} dz$$

where C is as in Figure 1.

Since e^{iz^2} is analytic on $\operatorname{Int} C$, Cauchy's Theorem gives us

$$\begin{aligned} 0 = I &= \int_{C_1} e^{iz^2} dz + \int_{C_2} e^{iz^2} dz + \int_{C_3} e^{iz^2} dz \\ &= \int_0^R e^{ix^2} dx + \int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} iRe^{i\theta} d\theta + \int_R^0 e^{i(xe^{i\frac{\pi}{4}})^2} e^{i\frac{\pi}{4}} dx. \end{aligned}$$

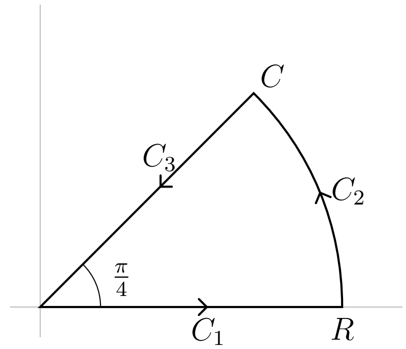


FIGURE 1

Equivalently, we have

$$0 = \int_0^R e^{ix^2}dx + iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta + iR^2\cos 2\theta}e^{i\theta}d\theta - e^{i\frac{\pi}{4}}\int_0^R e^{-x^2}dx. \tag{31.5}$$

Since

$$\lim_{R\to\infty}\int_0^R e^{-x^2}dx = \frac{\sqrt{\pi}}{2},$$

it follows from (31.5) that

$$\lim_{R\to\infty}\int_0^R e^{ix^2}dx = \frac{\sqrt{\pi}}{2}e^{\frac{i\pi}{4}} - \lim_{R\to\infty}\underbrace{iR\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta + iR^2\cos 2\theta}e^{i\theta}d\theta}_{:=I_R}. \tag{31.6}$$

We will now show that $I_R \to 0$ as $R \to \infty$. Indeed, since $\sin 2\theta \geq \frac{4}{\pi}\theta$ (recall from the proof of Jordan's lemma), we have

$$|I_R| \leq R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}d\theta \leq R\int_0^{\frac{\pi}{4}} e^{-\frac{4}{\pi}R^2\theta}d\theta = R\pi\frac{1-e^{-R^2}}{4R^2} \leq \frac{\pi}{4R}.$$

Thus, $\lim_{R\to\infty}|I_R| = 0$, and so (31.6) yields

$$\lim_{R\to\infty}\int_0^R e^{ix^2}dx = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}\frac{1+i}{\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}(1+i). \tag{31.7}$$

But

$$\int_0^R e^{ix^2}dx = \int_0^R \cos x^2 dx + i\int_0^R \sin x^2 dx$$

and hence (31.7) implies

$$\lim_{R\to\infty}\int_0^R \sin x^2 dx = \lim_{R\to\infty}\int_0^R \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

and consequently

$$F_1(a) = F_2(a) = \frac{1}{2}\sqrt{\frac{\pi}{2a}}. \tag{31.8}$$

REMARK 31.1. *What we actually have shown is that*

$$\int_{\mathbb{R}_+} e^{iz^2} dz = \int_{R_{\frac{\pi}{4}}} e^{iz^2} dz \tag{31.9}$$

where $R_{\frac{\pi}{4}}$ is, as before, the rotation of $\mathbb{R}_+$ by $\frac{\pi}{4}$. The integral on the right-hand side of (31.9)*, in turn, can be reduced to the Gauss integral $\int_0^\infty e^{-x^2} dx$.*

2. Indented Contours

In this subsection we consider an important Fourier integral

$$\int_{\mathbb{R}} e^{iax} f(x) dx \quad , \quad a > 0, \tag{31.10}$$

where $f(x)$ has a singularity on $\mathbb{R}$. We restrict ourselves to $f(x) = \frac{1}{x}$.

Consider the contour integral

$$I := \int_C \frac{e^{iaz}}{z} dz$$

where C is as in Figure 2.

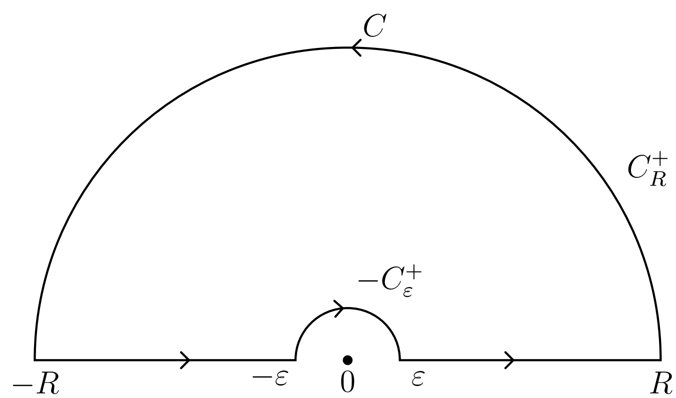


FIGURE 2

Note that $\frac{1}{x}$ does not satisfy Theorem 30.2 as $\frac{1}{z}$ has a simple pole on $\mathbb{R}$. To avoid 0 we have to make an indentation C_ε on our contour. Now $\frac{1}{z}$ is analytic on $\operatorname{Int} C$ and hence, by the Cauchy Theorem, we have

$$0 = \left(\underbrace{\int_{C_R^+}}_{I_1} - \underbrace{\int_{C_\varepsilon^+}}_{I_2} + \underbrace{\int_{(-R,R)\setminus[-\varepsilon,\varepsilon]}}_{I_3} \right) \frac{e^{iaz}}{z} dz. \tag{31.11}$$

Since $\dfrac{1}{z}$ is subject to Jordan's Lemma we have that

$$I_1 \to 0,\ R \to \infty. \tag{31.12}$$

For I_2, one has

$$\begin{aligned} I_2 &= \int_{C_\varepsilon^+} \frac{e^{iaz}}{z} dz = \int_0^\pi \frac{e^{ia\varepsilon(\cos\theta + i\sin\theta)}}{\varepsilon e^{i\theta}} i\varepsilon e^{i\theta} d\theta \\ &= i\int_0^\pi e^{ia\varepsilon(\cos\theta + i\sin\theta)} d\theta. \end{aligned}$$

This integral cannot be explicitly computed for $\varepsilon > 0$, but

$$I_2 \to i\pi\,,\ \varepsilon \to 0. \tag{31.13}$$

The proof of this limit is left as an exercise. From (31.10) we have

$$\int_{(-R,R)\setminus[-\varepsilon,\varepsilon]} \frac{e^{iax}}{x} dx = I_2 - I_1 \tag{31.14}$$

and taking the limit of (31.14) as $R \to \infty$ and $\varepsilon \to 0$, we have

$$\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{e^{iax}}{x} dx = \lim_{\varepsilon\to 0} I_2 - \lim_{R\to\infty} I_1 = i\pi. \tag{31.15}$$

The limit on the LHS of (31.15) is, by definition, the Cauchy principal value of $\int_{-\infty}^{\infty} \dfrac{e^{iax}}{x} dx$ and is typically denoted $\mathrm{V.P.}\int_{-\infty}^{\infty} f$. Thus (31.15) implies

$$\mathrm{V.P.}\int_{\mathbb{R}} \frac{e^{iax}}{x} dx = i\pi \quad , \quad a > 0. \tag{31.16}$$

If $a < 0$ then taking the complex conjugate of (31.16) one has

$$\mathrm{V.P.}\int_{\mathbb{R}} \frac{e^{iax}}{x} dx = -i\pi \quad , \quad a < 0. \tag{31.17}$$

Equations (31.16) and (31.17) imply

$$\mathrm{V.P.}\int_{\mathbb{R}} \frac{e^{iax}}{x} dx = i\pi \operatorname{sgn} a \tag{31.18}$$

$$\mathrm{V.P.}\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2} \operatorname{sgn} a. \tag{31.19}$$

Note that the integrals in (31.18) and (31.19) depend on the sign of a but not its value. The integral in (31.19) is understood as follows:

$$\mathrm{V.P.}\int_0^\infty f = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^R f.$$

Exercises

Exercise 31.1 Prove that for all $a > 0$

$$\lim_{\varepsilon \to 0} \int_{C_\varepsilon^+} \frac{e^{iaz}}{z} dz = i\pi.$$

Exercise 31.2 Show that

$$\text{V. P.} \int_{\mathbb{R}} \frac{1 - \cos x}{x^2} dx = \pi.$$

Exercise 31.3 Show that

$$\text{V. P.} \int_{\mathbb{R}} \frac{\sin ax}{x - b} dx = \pi \operatorname{sgn} a \cos ab.$$

LECTURE 32

The Art of Contour Integration: Integration Involving Branch Cuts

In this lecture we will be concerned with integrals involving root functions.

As in certain cases before, we are able to derive a general formula for integrals of the type: $\displaystyle\int_0^\infty x^\alpha f(x)dx$ for $0 < |\alpha| < 1$. But first we introduce a new concept:

DEFINITION 32.1. *A function $f(z)$ is called <u>meromorphic</u> if f is analytic on the whole of $\mathbb{C}$ except for poles.*

We have seen many meromorphic functions before. A rational function is meromorphic with a finite number of poles. $\tan z, \cot z$ are meromorphic with infinitely many simple poles. The function $e^{\frac{1}{z}}$ is not meromorphic as 0 is not a pole, but instead an essential singularity.

DEFINITION 32.2. *Let $f(x)$ be a function defined on an interval (a, b). We say that f admits a <u>meromorphic continuation</u> into $\mathbb{C}$ if there is a meromorphic function $f(z)$ such that $f(z)|_{(a,b)} = f(x)$.*

An example is $\tan x = \dfrac{\sin x}{\cos x}$ which admits a meromorphic continuation into $\mathbb{C}$ where $\tan z = \dfrac{\sin z}{\cos z}$ is its meromorphic continuation. Another example is the function $\dfrac{1}{x^2+1}$ which admits a meromorphic continuation $\dfrac{1}{z^2+1}$. On the other hand, the function $\sqrt{x}, x \in \mathbb{R}$, does not admit a meromorphic continuation as $\sqrt{z}$, because $\sqrt{z}$ cannot be defined as analytic on the whole of $\mathbb{C}$ (only on $\mathbb{C}$ with a suitable branch cut.)

For any $\alpha \in \mathbb{R}$ we define z^α as

$$z^\alpha \stackrel{\text{def}}{=} \exp \alpha \log z$$

with a sensible choice for the branch cut for $\log z$. E.g. we take the principal branch of $\log z$, i.e. define $\log z$ on $\mathbb{C} \setminus \mathbb{R}_+$ by

$$\log z \stackrel{\text{def}}{=} \log|z| + i \arg z, \qquad \arg z \in [0, 2\pi)$$

then z^α will be defined on $\mathbb{C} \setminus \mathbb{R}_+$ by

$$z^\alpha = e^{\alpha(\log|z| + i\arg z)} = |z|^\alpha e^{i\alpha \arg z}, \qquad 0 \le \arg z < 2\pi. \tag{32.1}$$

Note that if $\alpha \in \mathbb{Z}$ then the necessity of the branch cut disappears as

$$e^{i\alpha\cdot 0} = e^{i\alpha\cdot 2\pi}, \forall \alpha \in \mathbb{Z}.$$

This, of course, no longer holds when $\alpha = 1/n, n = 2, 3, \ldots$ Definition 32.2 turns into Definition 9.1 from Lecture 9.

Similarly to (32.1), one can define z^α on $\mathbb{C} \setminus \mathbb{R}_-$ or any other ray R_θ.

PROPOSITION 32.3. *Let a function $f(x), x \in \mathbb{R}_+$, have a meromorphic continuation $f(z)$. Assume $f(z)$ has a finite number of poles $\{z_k\}_{k=1}^n, z_k \notin \mathbb{R}_+$, and $zf(z)$ has a removable singularity at ∞. Then for any $\alpha \in (0,1)$*

$$\int_0^\infty x^{-\alpha} f(x)dx = \frac{2\pi i}{1 - e^{-2\pi i\alpha}} \sum_{k=1}^n \operatorname{Res}\{z^{-\alpha} f(z), z_k\} \tag{32.2}$$

where z^α is defined on $\mathbb{C} \setminus \mathbb{R}_+$.

PROOF. Introduce a new function $\varphi(z) = z^{-\alpha} f(z)$ where $z^{-\alpha}$ is defined on $\mathbb{C} \setminus \mathbb{R}_+$ by (32.1). The function $\varphi(z)$ is then analytic on $\mathbb{C} \setminus \mathbb{R}_+$ except at a finite number of poles $\{z_k\}_{k=1}^n$. Consider the integral $\displaystyle\int_C \varphi(z)dz$ where C is as in Figure 1.

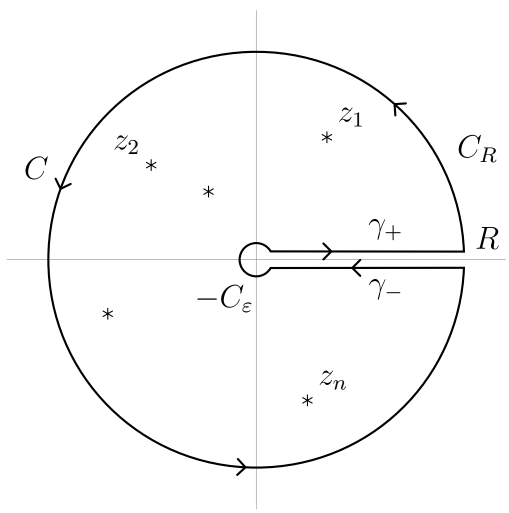


FIGURE 1

We have, by the Residue Theorem, for R large enough to include all z_k

$$2\pi i \sum_{k=1}^n \operatorname{Res}(\varphi, z_k) = \left(\underbrace{\int_{C_R}}_{I_1} + \underbrace{\int_{-C_\varepsilon}}_{I_2} + \underbrace{\int_{\gamma_+}}_{I_3} + \underbrace{\int_{\gamma_-}}_{I_4} \right) \varphi(z)dz. \tag{32.3}$$

As before, we take care of each I_k separately. By the conditions of the proposition we known that $zf(z)$ has a removable singularity at ∞. I.e. $\lim\limits_{z\to\infty} zf(z)$ exists and hence there exists $M > 0$ such that $|zf(z)| \leq M$ for any $|z| = R$. Hence for all $z \in C_R$

$$|\varphi(z)| = |z^{-\alpha}||f(z)| \leq |z|^{-\alpha} \cdot \frac{M}{|z|} = \frac{M}{R^{1+\alpha}}$$

and therefore

$$|I_1| \le \int_{C_R} |\varphi(z)||dz| \le \frac{M}{R^{1+\alpha}} 2\pi R = \frac{2\pi M}{R^\alpha} \to 0 \text{ as } R \to \infty. \tag{32.4}$$

For I_2 we have

$$\begin{aligned} |I_2| &\le \int_{C_\varepsilon} |\varphi(z)||dz| = \int_{C_\varepsilon} |z|^{-\alpha} |f(z)||dz| \\ &\le \underbrace{\sup_{|z|=\varepsilon} |f(z)|}_{\text{bounded as 0 is not a pole}} \cdot \underbrace{2\pi\varepsilon\varepsilon^{-\alpha}}_{=2\pi\varepsilon^{1-\alpha}} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{32.5}$$

For I_3, we have $I_3 = \int_\varepsilon^R x^{-\alpha} f(x)dx$. Now evaluate I_4. If $z \in \gamma_-$ then $z = xe^{2\pi i}, x > 0$ and by (32.1) $z^{-\alpha} = x^{-\alpha} e^{-2\pi i\alpha}$. Therefore

$$I_4 = \int_{\gamma_-} z^{-\alpha} f(z)dz = \int_R^\varepsilon x^{-\alpha} e^{-2\pi i\alpha} f(x)dx \;\; = -e^{-2\pi i\alpha} \underbrace{\int_\varepsilon^R x^{-\alpha} f(x)dx}_{I_3} = -e^{-2\pi i\alpha} I_3.$$

Now

$$I_3 + I_4 = (1 - e^{-2\pi i\alpha}) I_3 = (1 - e^{-2\pi i\alpha}) \int_\varepsilon^R x^{-\alpha} f(x)dx. \tag{32.6}$$

One the other hand, from (32.3)

$$I_3 + I_4 = 2\pi i \sum_{k=1}^n \operatorname{Res}(\varphi, z_k) - I_1 - I_2. \tag{32.7}$$

Combining (32.6) and (32.7) yields

$$\int_\varepsilon^R x^{-\alpha} f(x)dx = \frac{1}{1 - e^{-2\pi i\alpha}} \left(2\pi i \sum_{k=1}^n \operatorname{Res}(\varphi, z_k) - I_1 - I_2 \right). \tag{32.8}$$

Taking (32.8) to the limit as $R \to \infty, \varepsilon \to 0$, by (32.5) and (32.6) we finally have (32.2).

□

Example 32.4. *Evaluate* $I = \int_0^\infty \frac{dx}{x^\alpha (x+1)}, 0 < \alpha < 1$

Note that $\frac{1}{x+1}, x \ge 0$ *has a meromorphic continuation into* $\mathbb{C}$ *with one simple pole at* $z = -1$. *Also* $\lim_{z\to\infty} \frac{z}{z+1} = 1$ *and hence we are in the conditions of Proposition 32.3*

and by (32.2)

$$I = \frac{2\pi i}{1 - e^{-2\pi i\alpha}} \frac{1}{(-1)^\alpha} = \frac{2\pi i}{1 - e^{-2\pi i\alpha}} \frac{1}{e^{i\pi\alpha}}$$
$$= \frac{2\pi i}{e^{i\pi\alpha} - e^{-i\pi\alpha}} = \frac{\pi}{\sin \pi\alpha}.$$

The following proposition is also useful.

PROPOSITION 32.5. *Let a function $f(x), x \in \mathbb{R}_+$ have a meromorphic continuation $f(z)$. Assume $f(z)$ has a finite number of poles $\{z_k\}_{k=1}^n$ which are not on $\mathbb{R}_+$ and $z^2 f(z)$ has a removable singularity at ∞. Then for any $\alpha \in (0, 1)$*

$$\int_0^\infty x^\alpha f(x) dx = \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum_{k=1}^n \operatorname{Res}\{z^\alpha f(z), z_k\}. \tag{32.9}$$

The proof is left as an exercise.

Exercises

Exercise 32.1 Prove Proposition 32.5.

Exercise 32.2 Show that if $\alpha \in (-1, 1)$ then

$$\int_0^\infty \frac{x^\alpha}{(x+1)^2} dx = \frac{\pi\alpha}{\sin \pi\alpha}.$$

Exercise 32.3 Show that for any $\alpha \in (0, 1)$

$$\int_0^1 \frac{x^{\alpha-1}}{(1-x)^\alpha} dx = \frac{\pi}{\sin \pi\alpha}.$$

(Hint: Use the substitution $y = \dfrac{x}{1-x}$)

LECTURE 33

The Art of Contour Integration: Integrals of the Type $\int_0^\infty f(x)\log x dx$

In this lecture we derive a general formula for integrals of the type $\displaystyle\int_0^\infty f(x)\log x dx$.

PROPOSITION 33.1. *Let $f(x)$ be an even function which admits an analytic continuation into $\mathbb{C}^+$. Assume that $f(z)$ has finitely many poles $\{z_k\}_{k=1}^n$ in $\mathbb{C}^+$, no poles on $\mathbb{R}$, and for sufficiently large $R>0$ there exist $M,\delta>0$ such that*

$$|f(z)| \le \frac{M}{|z|^{1+\delta}} \quad , \quad \forall z : |z| \ge R. \tag{33.1}$$

Then

$$I := \int_0^\infty f(x)\log x dx = \pi i \sum_{k=1}^n \operatorname{Res}\left\{ f(z)\left(\log z - \frac{i\pi}{2}\right), z_k\right\}. \tag{33.2}$$

PROOF. We first introduce the function $\varphi(z) = f(z)\log z, z \in \mathbb{C}^+$, where $\log z$ is defined on $\mathbb{C}\setminus\mathbb{R}_+$. Consider the integral $\displaystyle\int_C \varphi(z)dz$ where C is as in Figure 1.

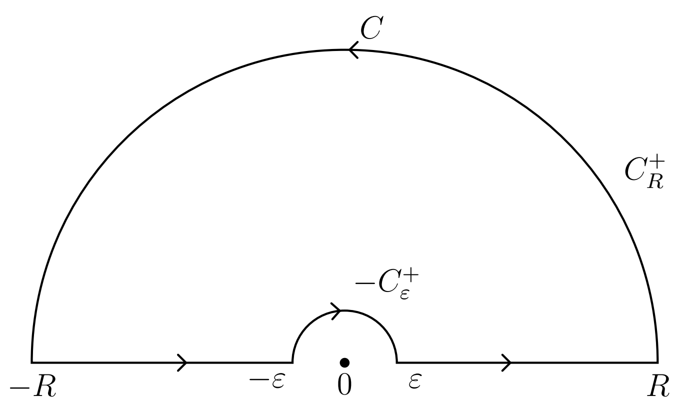


FIGURE 1

By the Residue Theorem,

$$2\pi i \sum_{k=1}^n \operatorname{Res}(\varphi, z_k) = \int_C \varphi(z)dz = \left(\underbrace{\int_{C_R^+}}_{:=I_1} + \underbrace{\int_{-C_\varepsilon^+}}_{:=I_2} + \underbrace{\int_\varepsilon^R}_{:=I_3} + \underbrace{\int_{-R}^{-\varepsilon}}_{:=I_4} \right)\varphi(z)dz. \tag{33.3}$$

For I_1 we have

$$|I_1| \leq \int_{C_R^+} |f(z)||\log z||dz| = \int_0^\pi \left|f(Re^{i\theta})\right| |\log R + i\theta| R d\theta$$
$$\leq \int_0^\pi \frac{M}{R^{1+\delta}}(\log R + \pi) R d\theta \leq \frac{\pi M(\log R + \pi)}{R^\delta} \overset{\text{L'Hôpital}}{\longrightarrow} 0 \quad , \quad R \to \infty. \tag{33.4}$$

For I_2 one obtains

$$|I_2| \to 0 \quad , \quad \varepsilon \to 0. \tag{33.5}$$

The proof of (33.5) is left as an exercise.

For I_3,

$$I_3 = \int_\varepsilon^R f(x) \log x dx \to I \quad , \quad R \to \infty, \varepsilon \to 0. \tag{33.6}$$

Now all that remains is to consider I_4. On $(-R, -\varepsilon)$, $z = xe^{i\pi}$ where $x > 0$, so we have

$$\varphi(z) = \varphi(xe^{i\pi}) = f(-x) \log(xe^{i\pi}) \overset{\text{f is even}}{=} f(x)(\log x + i\pi).$$

Thus

$$I_4 = \int_{-R}^{-\varepsilon} \varphi(z) dz = \int_\varepsilon^R f(x)(\log x + i\pi) dx = I_3 + i\pi \int_\varepsilon^R f(x) dx. \tag{33.7}$$

Since we have required $f(z)$ to have no poles on $\mathbb{R}$ and $f(x)$ is even, we have

$$\lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \int_\varepsilon^R f(x) dx = \int_0^\infty f(x) dx = \frac{1}{2} \int_{\mathbb{R}} f(x) dx. \tag{33.8}$$

But f satisfies (33.1) and so $\int_{\mathbb{R}} f(x)dx$ was evaluated in Theorem 28.7:

$$\int_{\mathbb{R}} f(x) dx = 2\pi i \sum_{k=1}^n \operatorname{Res}(f, z_k). \tag{33.9}$$

Now by (33.6) and (33.7),

$$I_3 + I_4 = 2I_3 + i\pi \int_\varepsilon^R f(x) dx \tag{33.10}$$

and hence we have

$$\lim_{\substack{R\to\infty \\ \varepsilon \to 0}} (I_3 + I_4) = 2 \lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \int_\varepsilon^R f(x) \log x dx + i\pi \lim_{\substack{R\to\infty \\ \varepsilon \to 0}} \int_\varepsilon^R f(x) dx$$
$$\overset{(33.9)}{=} 2 \int_0^\infty f(x) \log x dx + (i\pi)^2 \sum_{k=1}^n \operatorname{Res}(f, z_k).$$

Thus

$$I_3 + I_4 \to 2I + (i\pi)^2 \sum_{k=1}^n \operatorname{Res}(f, z_k) \quad , \quad R \to \infty, \varepsilon \to 0. \tag{33.11}$$

We now go back to (33.3) and let $R \to \infty$, $\varepsilon \to 0$. By virtue of (33.4), (33.5), and (33.11),

$$2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z)\log z, z_k) = 2I + (i\pi)^2 \sum_{k=1}^{n} \operatorname{Res}(f, z_k). \tag{33.12}$$

Solving for I we arrive at (33.2). $\square$

Example 33.2. *Evaluate*

$$I = \int_0^\infty \frac{\log x}{1+x^2} dx.$$

Note that $f(x) := \dfrac{1}{1+x^2}$ *satisfies* (33.1) *and has one pole in* $\mathbb{C}^+$ *at* i. *Thus, by* (33.2)

$$I = \pi i \operatorname{Res}\left\{\frac{1}{1+z^2}\left(\log z - \frac{i\pi}{2}\right), i\right\} = \pi i \frac{\log i - i\frac{\pi}{2}}{2i} = 0.$$

Exercises

Exercise 33.1 Prove (33.5).

Exercise 33.2 Show that

$$\int_0^\infty \frac{\log^2 x}{x^2+1} dx = \frac{\pi^3}{8}.$$

Exercise 33.3 Prove that

$$\int_0^1 \frac{\log^2 x}{x^2+1} dx = \int_1^\infty \frac{\log^2 x}{x^2+1} dx = \frac{\pi^3}{16}.$$

LECTURE 34

The Art of Contour Integration: Integrals of the Type $\int_0^1 x^{\alpha-1}(1-x)^{-\alpha}f(x)dx$

The main obstacle here to evaluating integrals of the type $\displaystyle\int_0^1 z^{\alpha-1}(1-z)^{-\alpha}f(z)dz$ is making a suitable branch cut in order to define $z^{\alpha-1}(1-z)^{-\alpha}$.

PROPOSITION 34.1. *Let a function $f(x)$, $x \in (0,1)$, have a meromorphic continuation $f(z)$. Assume $f(z)$ has a finite number of poles $\{z_k\}_{k=1}^n$ such that $z_k \notin [0,1]$, and $f(z)$ has a removable singularity at ∞. Then for any $\alpha \in (0,1)$*

$$I := \int_0^1 \frac{x^{\alpha-1}}{(1-x)^\alpha} f(x)dx = \frac{\pi f(\infty)}{\sin \pi\alpha} + \frac{2\pi i}{1-e^{2\pi i\alpha}} \sum_{k=1}^n \operatorname{Res}\left\{\frac{z^{\alpha-1}}{(1-z)^\alpha} f(z), z_k\right\}. \tag{34.1}$$

PROOF. We introduce a new function

$$\varphi(z) = \left(\frac{z}{1-z}\right)^\alpha \frac{f(z)}{z}.$$

The main issue is how to define $\left(\dfrac{z}{1-z}\right)^\alpha$, which in turn will determine our choice of the contour C.

Let's define z^α on $\mathbb{C} \setminus \mathbb{R}_+$ the same way as in Lecture 32:

$$z^\alpha = |z|^\alpha e^{i\alpha\theta_1}, \quad \theta_1 := \arg z \in [0, 2\pi). \tag{34.2}$$

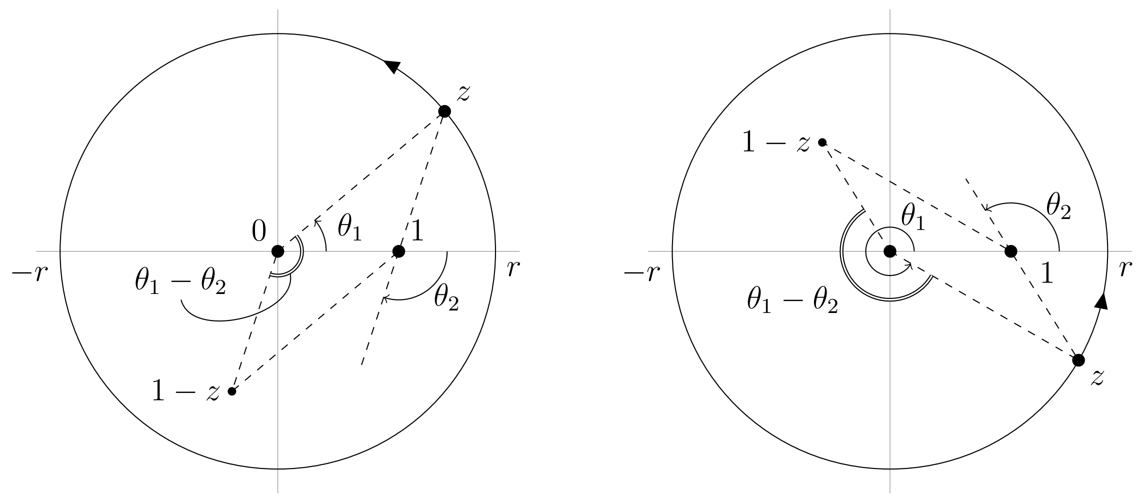
Similarly, define $(1-z)^\alpha$ on $\mathbb{C} \setminus [1, \infty)$ by

$$(1-z)^\alpha = |1-z|^\alpha e^{i\alpha\theta_2}, \quad \theta_2 := \arg(1-z) \in [-\pi, \pi).$$

We now define

$$\left(\frac{z}{1-z}\right)^\alpha = \frac{z^\alpha}{(1-z)^\alpha} = \left|\frac{z}{1-z}\right|^\alpha e^{i\alpha(\theta_1-\theta_2)} \tag{34.3}$$

We show that $\left(\dfrac{z}{1-z}\right)^\alpha$ defined by (34.3) is analytic on $\mathbb{C} \setminus [0,1]$. One can easily see that $\left(\dfrac{z}{1-z}\right)^\alpha$ is analytic on at least $\mathbb{C} \setminus \mathbb{R}_+$ and hence it's left to demonstrate that $\left(\dfrac{z}{1-z}\right)^\alpha$ is continuous across $(1, \infty)$. To this end let's go around the origin along the circle $C_r(0)$, $r > 1$ as in Figure 1(a) and Figure 1(b).



(a) $z \in \mathbb{C}^+,\ 0 \leq \theta_1 \leq \pi,\ \theta_2 < 0$ (b) $z \in \mathbb{C}^-,\ \pi < \theta_1 < 2\pi,\ \theta_2 > 0$

FIGURE 1

Now consider the exponential $e^{i\alpha(\theta_1 - \theta_2)}$ on the right hand side of (34.3) as a function of $\arg z$ where $\arg z \in [0, 2\pi)$. One can easily draw a graph of $\theta_1 - \theta_2$ as a function of $\arg z$. For your convenience we have included this picture in Figure 2.

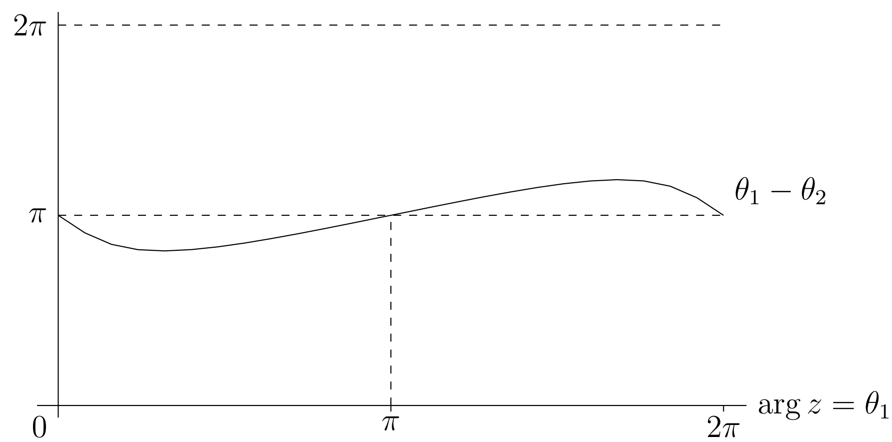


FIGURE 2

It follows from Figure 2 that $e^{i\alpha(\theta_1 - \theta_2)}$ is a continuous function of $\arg z$ having the same value at 0 and 2π. (So called <u>continuous on the circle</u>). Therefore, $\left(\dfrac{z}{1-z}\right)^\alpha$ is analytic on $\mathbb{C} \setminus [0, 1]$.

We now evaluate $\left(\dfrac{z}{1-z}\right)^\alpha$ on the upper (lower) γ_+ (γ_-) bank of the branch cut $[0, 1]$. Now look at Figure 3.

Consider as above the exponent $e^{i\alpha(\theta_1 - \theta_2)}$ on the right hand side of (34.3) as a function of z running along $C_r(0)$ with some $r < 1$. (Recall, in our previous case we

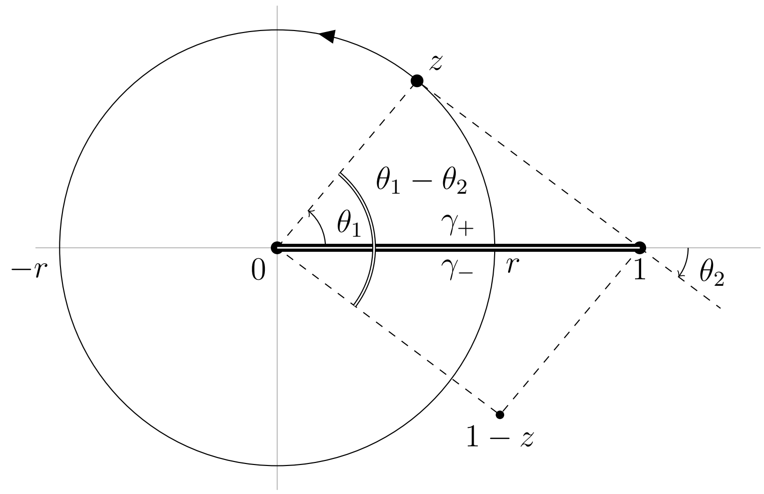


FIGURE 3

chose $r > 1$.) The graph of $\theta_1 - \theta_2$ as a function of $\arg z \in [0, 2\pi)$ is given in Figure 4. (Before reading on make sure that you understand this graph.)

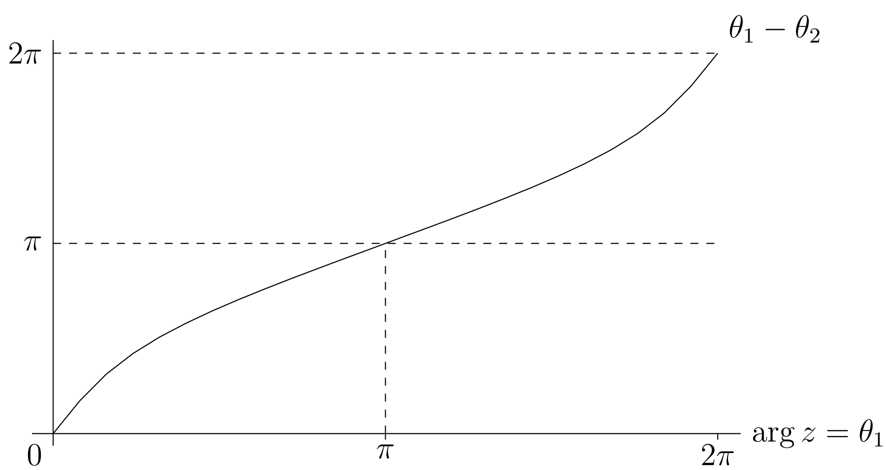


FIGURE 4

It follows from Figure 4 that $e^{i\alpha(\theta_1-\theta_2)}$ is no longer continuous on the unit circle since

$$e^{i\alpha(\theta_1-\theta_2)}\Big|_{\gamma_+} = e^{i\alpha 0} = 1$$

and

$$e^{i\alpha(\theta_1-\theta_2)}\Big|_{\gamma_-} = e^{2i\alpha\pi} \neq 1.$$

Hence $e^{i\alpha(\theta_1-\theta_2)}$ has a jump discontinuity across $[0, 1]$.

Thus, by (34.3),

$$z = xe^{i0} \in \gamma_+ \implies \left(\frac{z}{1-z}\right)^\alpha = \left(\frac{x}{1-x}\right)^\alpha \tag{34.4}$$

$$z = xe^{2\pi i} \in \gamma_- \implies \left(\frac{z}{1-z}\right)^\alpha = \left(\frac{x}{1-x}\right)^\alpha e^{2\pi i\alpha}. \tag{34.5}$$

Note that (34.4) and (34.5) show that $\left(\frac{z}{1-z}\right)^\alpha$ is discontinuous across the branch cut $[0,1]$.

We have now defined $\left(\frac{z}{1-z}\right)^\alpha$ as an analytic function on $\mathbb{C}\setminus[0,1]$ and are ready to construct our contour C.

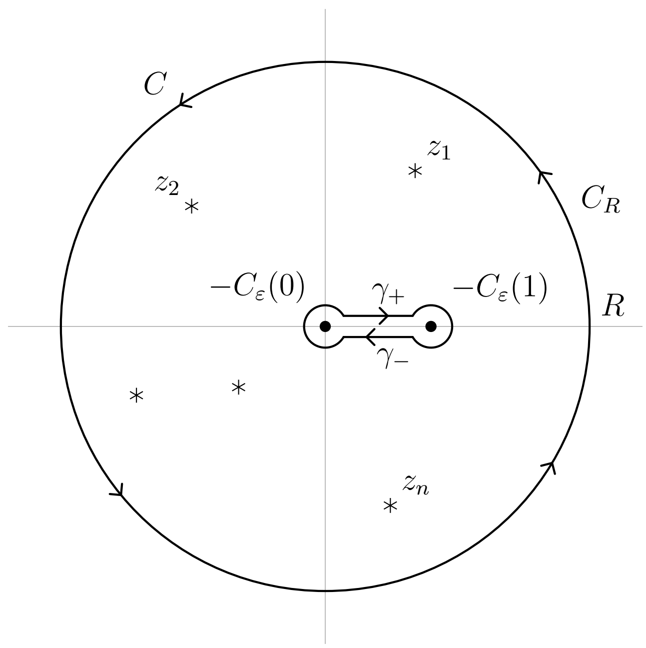


Figure 5

Take C as in Figure 5 and form the integral

$$\int_C \varphi(z)dz.$$

We choose R large enough to enclose all the poles $\{z_k\}_{k=1}^n$. By the Residue Theorem

$$\begin{aligned}2\pi i\sum_{k=1}^n \operatorname{Res}(\varphi, z_k) = \int_C \varphi(z)dz &= \left(\int_{C_R} - \int_{C_\varepsilon(0)} - \int_{C_\varepsilon(1)} + \int_{\gamma_+} + \int_{\gamma_-}\right)\varphi(z)dz \\ &= I_1 - I_2 - I_3 + I_4 + I_5. \end{aligned} \tag{34.6}$$

As opposed to all of our previous integrals $I_1 \not\to 0$ as $R\to\infty$, but it can be evaluated explicitly. Indeed $\varphi(z)$ is analytic on $\operatorname{Ext} C_R$ and hence by Theorem 26.14

we have

$$\int_{C_R} \varphi(z)dz = -2\pi i \operatorname{Res}(\varphi, \infty) = 2\pi i a_{-1}$$

where a_{-1} is the corresponding Laurent coefficient.

We now find a_{-1} explicitly. We have

$$\lim_{z\to\infty} \left(\frac{z}{1-z}\right)^{\alpha} = (-1)^{\alpha} = e^{\alpha\pi i} \tag{34.7}$$

and hence

$$\begin{aligned} a_{-1} &= \lim_{z\to\infty} z\varphi(z) = \lim_{z\to\infty} \left(\frac{z}{1-z}\right)^{\alpha} f(z) \\ &= \underbrace{\lim_{z\to\infty} \left(\frac{z}{1-z}\right)^{\alpha}}_{=e^{\alpha\pi i} \text{ by (34.7)}} \cdot \underbrace{\lim_{z\to\infty} f(z)}_{=f(\infty) \text{ by assumption}} \\ &= e^{\alpha\pi i} f(\infty). \end{aligned}$$

Thus,

$$I_1 = 2\pi i f(\infty) e^{\pi i\alpha}. \tag{34.8}$$

Next, one can easily see that (though the details are left as an exercise) that

$$\lim_{\varepsilon\to 0} I_2 = \lim_{\varepsilon\to 0} I_3 = 0. \tag{34.9}$$

For I_4 we have

$$\begin{aligned} I_4 &= \int_{\gamma_+} \frac{1}{z}\Big(\frac{z}{1-z}\Big)^{\alpha} f(z)dz \overset{(34.5)}{=} \int_{\varepsilon}^{1-\varepsilon} \frac{1}{x}\Big(\frac{x}{1-x}\Big)^{\alpha} \frac{f(x)}{x} dx \\ &= \int_{\varepsilon}^{1-\varepsilon} \frac{x^{\alpha-1}}{(1-x)^{\alpha}} f(x)dx \ \to \ I, \text{ as } \varepsilon \to 0. \end{aligned} \tag{34.10}$$

For I_5 we have

$$\begin{aligned} I_5 &= \int_{\gamma_-} \frac{1}{z}\Big(\frac{z}{1-z}\Big)^{\alpha} f(z)dz \overset{(34.5)}{=} \int_{1-\varepsilon}^{\varepsilon} \frac{1}{x}\Big(\frac{x}{1-x}\Big)^{\alpha} e^{2\pi i\alpha} f(x)dx \\ &= -e^{2\pi i\alpha} \int_{\varepsilon}^{1-\varepsilon} \frac{x^{\alpha-1}}{(1-x)^{\alpha}} f(x)dx \ \to \ -e^{2\pi i\alpha} I \text{ as } \varepsilon \to 0. \end{aligned} \tag{34.11}$$

Taking $\varepsilon \to 0$ in (34.6) (Note that R doesn't have to go to ∞!) and using (34.8) - (34.11) we get

$$\begin{aligned} 2\pi i \sum_{k-1}^{n} \operatorname{Res}(\varphi, z_k) &= 2\pi i f(\infty) e^{\pi i\alpha} + I - e^{2\pi i\alpha} I \\ &= 2\pi i f(\infty) e^{\pi i\alpha} + (1 - e^{2\pi i\alpha}) I. \end{aligned}$$

Solving for I,

$$I = \frac{\pi f(\infty)}{\sin \pi\alpha} + \frac{2\pi i}{1 - e^{2\pi i\alpha}} \sum_{k=1}^{n} \operatorname{Res}(\varphi, z_k)$$

and (34.1) follows. □

EXAMPLE 34.2. *If* $\alpha \in (0,1)$ *then*

$$I = \int_0^1 x^{\alpha-1}(1-x)^{-\alpha}dx = \frac{\pi}{\sin \pi\alpha}.$$

REMARK 34.3. *Example 34.2 was offered before as an exercise. As a matter of fact the* I *in Proposition 34.1 could be reduced to*

$$\int_0^\infty \frac{f(x)}{x^\alpha}dx, \quad 0 < \alpha < 1,$$

which was evaluated before. However, it's generally not a good idea to approach this problem in this manner.

Exercises

Exercise 34.1 Prove (34.9), i.e. that $\lim\limits_{\varepsilon\to 0} I_2 = \lim\limits_{\varepsilon\to 0} I_3 = 0$.

Exercise 34.2 The beta function of two variables p, q is defined as

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt.$$

Show that

$$B(p, 1-p) = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1.$$

LECTURE 35

Logarithmic Residue And The Argument Principle

In this lecture we apply the Residue Theorem to counting zeros and poles of analytic functions.

1. Logarithmic Residue

DEFINITION 35.1. *Let $f(z)$ be analytic on some neighborhood of z_0 and $f(z_0) = 0$. We say that z_0 is an* <u>*m order zero*</u> *of $f(z)$ if*

$$f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$$

but $f^{(m)}(z_0) \neq 0$. The number m is called the <u>*multiplicity*</u> *of z_0.*

In parallel to the definition of isolated singularity, we say that z_0 is an isolated zero of $f(z)$ if there is a $\delta > 0$ such that for all $z \in \mathring{\mathbb{D}}_\delta(z_0)$ then $f(z) \neq 0$.

PROPOSITION 35.2. *Assume $f(z)$ is analytic on E and $z_0 \in E$ is a zero of f. Then*

(1) *z_0 has finite multiplicity.*
(2) *$f(z) = (z - z_0)^m \varphi(z)$ where φ is analytic at z_0 and $\varphi(z_0) \neq 0$.*

The proof is left as an exercise.

REMARK 35.3. *Proposition 35.2 says that there is a big difference between zeros of analytic functions and non-analytic functions. Indeed, consider the real function:*

$$\forall x \in \mathbb{R} : \qquad f(x) = \begin{cases} e^{-\frac{1}{x^2}}, x \neq 0 \\ 0, x = 0. \end{cases} \tag{35.1}$$

One can show (do it!) that $x = 0$ is a removable point of discontinuity of $f^{(n)}(x)$ for all $n \in \mathbb{N}$. Thus 0 is a zero of <u>*infinite multiplicity*</u> *of $f(x)$. By Proposition 35.2 this never happens to zeros of analytic functions.*

DEFINITION 35.4. *Let f be analytic. Then*

$$f'(z)/f(z)$$

is called the <u>*logarithmic derivative*</u>*.*

Here is a simple but important Lemma.

LEMMA 35.5 (The Logarithmic Residue Lemma). *Let $f(z)$ be analytic on some $\mathring{\mathbb{D}}_\delta(z_0)$(i.e. it is analytic on some neighborhood of z_0.) Then*

(1) *z_0 is an order m zero* $\Rightarrow \operatorname{Res}(f'(z)/f(z), z_0) = m$
(2) *z_0 is an order p pole* $\Rightarrow \operatorname{Res}(f'(z)/f(z), z_0) = -p$

(3) z_0 *is a removable singularity* $\Rightarrow \operatorname{Res}(f'(z)/f(z), z_0) = 0$.

PROOF.

(1) By Proposition 35.2 we have $f(z) = (z - z_0)^m \varphi(z)$ with $\varphi(z)$ analytic at z_0 and $\varphi(z_0) \neq 0$, then

$$\frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1}\varphi(z) + (z - z_0)^m \varphi'(z)}{(z - z_0)^m \varphi(z)} = \frac{m}{z - z_0} + \frac{\varphi'(z)}{\varphi(z)}. \tag{35.2}$$

Since $\varphi(z_0) \neq 0$ there exists $\delta : \varphi(z) \neq 0$ for all $z \in \mathbb{D}_\delta(z_0)$, the function $\varphi'(z)/\varphi(z)$ is analytic on $\mathbb{D}_\delta(z_0)$. It follows from (35.2) then z_0 is a simple pole of $f'(z)/f(z)$ with residue m. Part (1) has now been proved.

(2) Similarly to part (1), by Theorem 26.3 we have $f(z) = (z - z_0)^{-p}\varphi(z)$ with $\varphi(z)$ analytic at z_0 and $\varphi(z_0) \neq 0$ and

$$\frac{f'(z)}{f(z)} = \frac{-p}{z - z_0} + \frac{\varphi'(z)}{\varphi(z)}$$

and part (2) follows.

(3) Is trivial since we can think of a removable singularity as a pole of order zero.

□

COROLLARY 35.6 (The Counting Principle). *Let f be analytic on E except for poles and C be a closed simple contour in E with no zeros or poles on it. If C encloses n zeros and p poles of f then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P \tag{35.3}$$

where

$$N = \sum_{k=1}^{n} m_k, \qquad P = \sum_{k=1}^{p} p_k$$

and $m_k(p_k)$ is, in turn, the order of the k^{th} zero (k^{th} pole) in $\operatorname{Int} C$.

The proof will be left as an exercise.

2. The Argument Principle

Let $f(z)$ be a continuous function on a contour $C_{z_0z_1}$ and $f(z) \neq 0$ on $C_{z_0z_1}$ (i.e. $f(z)$ has no zeros on $C_{z_0z_1}$). One can write $f(z)$ in exponential form

$$f(z) = |f(z)|e^{i\theta(z)}. \tag{35.4}$$

Since $f(z)$ is continuous we can choose the function $\theta(z)$ to be continuous on $C_{z_0z_1}$. Set

$$\operatorname{Var}\arg_{C_{z_0z_1}} f \stackrel{\text{def}}{=} \theta(z_1) - \theta(z_0)$$

which is called the variation of the argument of f along $C_{z_0z_1}$.

One can easily see that Var arg does not depend on the choice of $\theta(z)$ and is determined by the function and contour.

Example 35.7. *Let* $f(z) = z^3$ *and* $C_{1,-1} = C_1^+(0)$. *Then*

$$f(z) = \underbrace{|z^3|}_{=1} e^{i\theta(z)} = e^{i\theta(z)}, z \in C_{1,-1}.$$

Choose $\theta(1) = 0$. *Then*

$$\theta(z) = 3\arg z, \arg z \in [0, 2\pi) \tag{35.5}$$

and hence $\theta(-1) = 3\arg(-1) = 3\pi$. *Thus,*

$$\operatorname{Var}\arg_{C_{1,-1}} z^3 = \theta(-1) - \theta(1) = 3\pi.$$

Example 35.8. *Let* $f(z) = z^3$ *and*

(1) $C = C_1(0)$. *Then by* (35.5)

$$\operatorname{Var}\arg_{C_1(0)} z^3 = 6\pi,$$

which means that $f(C_1(0))$ *is a contour (not simple) going around* 0 *three times.*

(2) $C = C_1(2)$. *Then to make* $\theta(z)$ *continuous we let* $\arg z \in [-\pi, \pi)$. *We have*

$$\operatorname{Var}\arg_{C_1(2)} z^3 = 0.$$

(Draw a picture and see it for yourself.)

Lemma 35.9. *Let* f_1, f_2 *be analytic on a simple contour* C *and* f_1, f_2 *have no zeros on* C. *Then*

(1)

$$\operatorname{Var}\arg_C f_1 f_2 = \operatorname{Var}\arg_C f_1 + \operatorname{Var}\arg_C f_2$$

(2)

$$\operatorname{Var}\arg_C \left(\frac{f_1}{f_2}\right) = \operatorname{Var}\arg_C f_1 - \operatorname{Var}\arg_C f_2.$$

The proof will be left as an exercise.

Theorem 35.10 (The Argument Principle). *Let* f *and* C *be as in Corollary 35.6 (i.e. let* f *be analytic on* E *except for poles and* C *be a closed simple contour in* E *with no zeros or poles on it.) Then*

$$\frac{1}{2\pi}\operatorname{Var}\arg_C f = N - P$$

where N *and* P *are as in Corollary 35.6 (i.e.* $N = \sum_{k=1}^{n} m_k, P = \sum_{k=1}^{p} p_k$.*)*

Proof. It follows from (35.3) that the integral on the left-hand side of (35.3) is independent of C and we can make it smooth. We only need to show now that

$$i\operatorname{Var}\arg_C f = \int_C \frac{f'(z)}{f(z)} dz. \tag{35.6}$$

But if C is smooth, we can parametrize it. That is, there is a smooth function $\gamma(t)$

$$\gamma : [0,1] \to C$$

such that $\gamma'(t) \neq 0$ and $\gamma(0) = \gamma(1) = z_0$, a fixed point on C. By (14.2) We have

$$\int_C \frac{f'(z)}{f(z)} dz = \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) dt. \tag{35.7}$$

On the other hand

$$f(\gamma(t)) = r(t) e^{i\phi(t)},$$

where $r(t) = |f(\gamma(t))|$ and $\phi(t) = \theta(\gamma(t))$, where θ is defined as in (35.4).

$$\begin{aligned} \frac{[f(\gamma(t))]'}{f(\gamma(t))} &= \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))} = \frac{r'(t)e^{i\phi(t)} + ir(t)e^{i\phi(t)}\phi'(t)}{r(t)e^{i\phi(t)}} \\ &= \frac{r'(t)}{r(t)} + i\phi'(t) = \frac{d}{dt} \ln r(t) + i\phi'(t). \end{aligned} \tag{35.8}$$

Plugging (35.8) into (35.7) we get:

$$\begin{aligned} \int_C \frac{f'(z)}{f(z)} dz &= \int_0^1 \frac{d}{dt} (\ln r(t) + i\phi(t)) dt \\ &= \underbrace{\ln r(1) - \ln r(0)}_{= \ln|f(z_0)| - \ln|f(z_0)| = 0} + i \underbrace{\phi(1) - \phi(0)}_{\mathrm{Var}\arg_C f} = i \,\mathrm{Var}\arg_C f, \end{aligned}$$

and (35.6) is proven. □

REMARK 35.11. *The Argument Principle is hard to use!*

REMARK 35.12. *Figure 1 below illustrates the Argument Principle for $N = 5$, $P = 3$.*

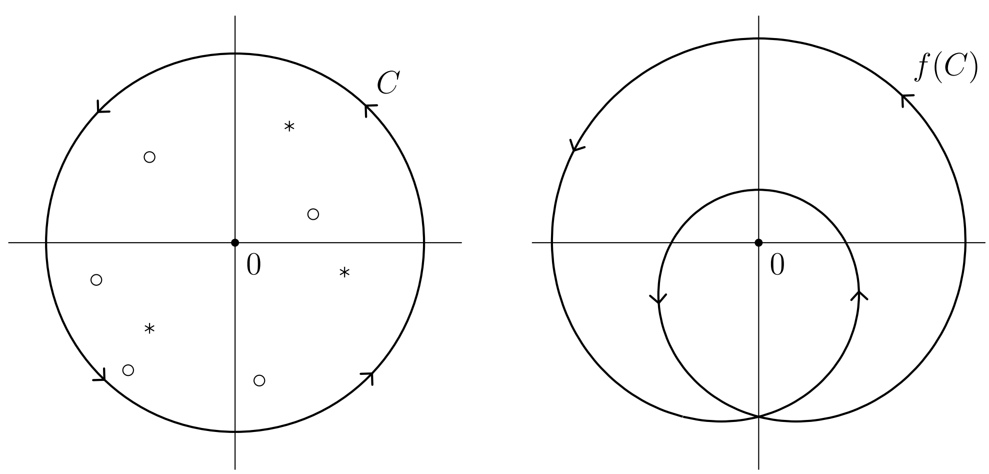


FIGURE 1

Exercises

Exercise 35.1 Prove Proposition 35.2.

Exercise 35.2 Prove every statement in Remark 35.3 and argue what happens to $f(z)$ if we allow x in (35.1) to be complex.

Exercise 35.3 Prove Corollary 35.6.

Exercise 35.4 Prove Lemma 35.9.

LECTURE 36

The Rouché Theorem

The Argument Principle is hard to implement in reality. The following statement has much more value in application to zero counting.

THEOREM 36.1 (Rouché's Theorem). *Let f and g be analytic inside and on a closed simple contour C (i.e., analytic inside C and at every point of C).*

If

$$|f(z)| > |g(z)| \quad \forall z \in C, \tag{36.1}$$

then inside C,

$$N(f+g) = N(f).$$

PROOF. By the Triangle Inequality,

$$|f(z)+g(z)| \geq \underbrace{|f(z)| - |g(z)|}_{>0 \text{ by } (36.1)} \quad \forall z \in C.$$

Hence the function $f(z)+g(z)$ has no zeros on C. Due to (36.1), $f(z)$ has no zeros on C either. By the Argument Principle,

$$\begin{aligned} N(f+g) - N(f) &= \frac{1}{2\pi} \operatorname{Var} \arg_C(f+g) - \frac{1}{2\pi} \operatorname{Var} \arg_C(f) \\ &\overset{\text{Lemma 35.9}}{=} \frac{1}{2\pi} \operatorname{Var}\ \arg_{\mathrm{C}} \frac{f+g}{f} \\ &= \frac{1}{2\pi} \operatorname{Var} \arg_C(1+\frac{g}{f}). \end{aligned} \tag{36.2}$$

Consider the function $\varphi(z) = 1 + g(z)/f(z)$. We have

$$|\varphi(z) - 1| = |g(z)|/|f(z)| \overset{(36.1)}{<} 1 \quad \forall z \in C.$$

Thus $|\varphi(z) - 1| < 1 \quad \forall z \in C$, which means that the contour $\varphi(C)$ is in the disk $\mathbb{D}_\rho(1)$ for some $\rho < 1$ and $0 \notin \mathbb{D}_\rho(1)$, as shown in Figure 1.

Hence $f(C)$ does not go around the origin and so $\operatorname{Var}\arg_C(1+g/f) = 0$. Thus the right hand side of (36.2) is 0 and Rouché's Theorem is proven.

$\square$

REMARK 36.2. *Rouché's Theorem tells us, for an analytic function f in a given region, a sufficiently small perturbation will not change the number of zeros of f counting multiplicity.*

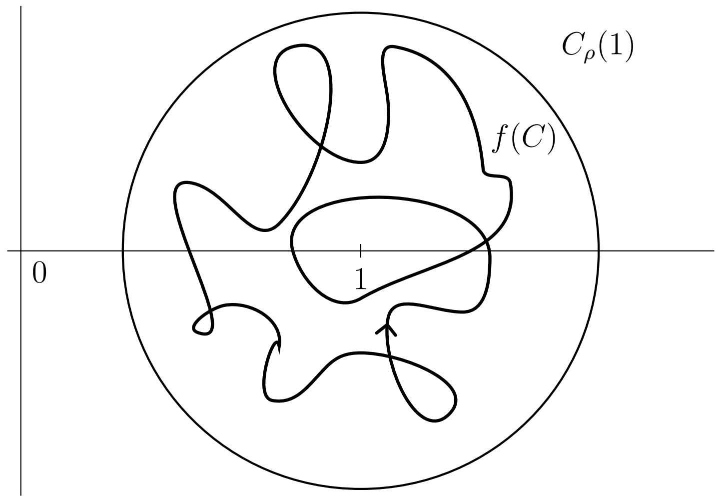


FIGURE 1

EXAMPLE 36.3. *Find the total number of zeros (N) of $F(z) = z^8 - 5z^5 - 2z + 1$ inside $\mathbb{D}$.*

Solution We have

$$F(z) = \underbrace{-5z^5 + 1}_{=f(z)} + \underbrace{z^8 - 2z}_{=g(z)}.$$

For all $|z| = 1$, we have

$$|f(z)| = |-5z^5 + 1| \geq 5|z|^5 - 1 = 4,$$

$$|g(z)| = |z^8 - 2z| \leq |z|^8 + 2|z| = 3.$$

Thus

$$|f(z)| > |g(z)| > 0 \qquad \forall z \in C_1(0).$$

Hence, by Rouché's Theorem, inside $C_1(0)$, $N(F) = N(f + g) = N(f)$. By the Fundamental Theorem of Algebra,

$$N(f) = N(-5z^5 + 1) = 5,$$

and hence $N = 5$.

Exercises

Exercise 36.1 Derive the Fundamental Theorem of Algebra from Rouché's Theorem.

Exercise 36.2 Find the number of zeros of $z^3 + 3z + 1$ inside $C_1(0)$.

Exercise 36.3 Find the number of zeros of $e^{z^2} - 4z^2$ inside $C_1(0)$.

Exercise 36.4 Find $\operatorname{Var}\arg_{C_1(0)} \dfrac{z^3+2}{z}$.

LECTURE 37

The Art of Contour Integration: Sums for some Series

Our last application of the Residue Theorem is related to the exact evaluation of numerical series like

$$\sum_{n\geq 1}\frac{1}{n^2} \quad , \quad \sum_{n\geq 1}\frac{1}{n^4}.$$

You probably remember from calculus that these series can be evaluated exactly. Now we are going to understand how. The approach is based on the following elementary lemma.

LEMMA 37.1. *Consider f analytic at $z\in\mathbb{Z}$. Then*

$$\operatorname{Res}\{f(z)\cot\pi z, n\} = \frac{1}{\pi}f(n). \tag{37.1}$$

PROOF. Since $\cot\pi z$ has simple poles at any integer,

$$\operatorname{Res}\{f(z)\cot\pi z, n\} = \lim_{z\to n}\frac{f(z)\cos\pi z}{(\sin\pi z)'} = \frac{f(n)}{\pi}.$$

□

Our main statement is

THEOREM 37.2. *Let $f(z)=p(z)/q(z)$ be a rational function and* $\deg q - \deg p \geq 2$. *If $f(z)$ has no integer poles then*

$$\sum_{-\infty}^{\infty} f(n) = -\pi\sum_{k=1}^{K}\operatorname{Res}\{f(z)\cot\pi z, z_k\} \tag{37.2}$$

where $\{z_k\}_{k=1}^{K}$ are the poles of f.

PROOF. Consider the contour C_N in Figure 1.

Choose N large enough to enclose all poles of $f(z)$. By the Residue Theorem

$$I_N := \int_{C_N} f(z)\cot\pi z dz = 2\pi i\left[\sum_{n=-N}^{N}\underbrace{\operatorname{Res}\left(f(z)\cot\pi z, n\right)}_{=f(n)/\pi \text{ by } (37.1)} + \sum_{k=1}^{K}\operatorname{Res}\left(f(z)\cot\pi z, z_k\right)\right]$$

$$= 2i\sum_{n=-N}^{N} f(n) + 2\pi i\sum_{k=1}^{K}\operatorname{Res}\left(f(z)\cot\pi z, z_k\right). \tag{37.3}$$

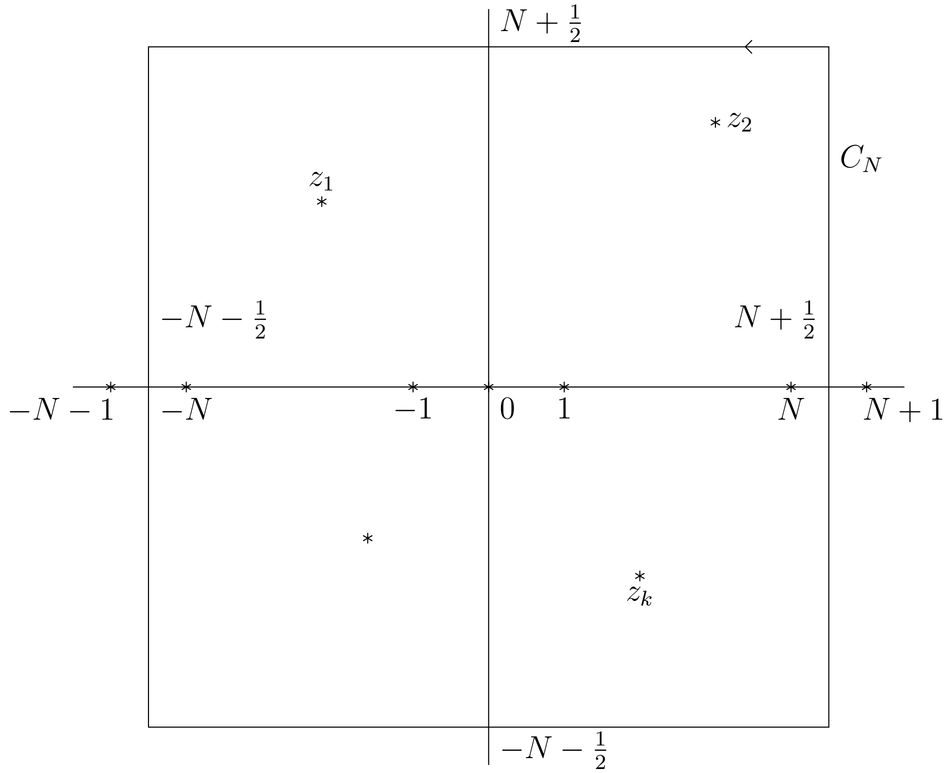


FIGURE 1

It's left to show that $I_N \to 0$ as $N \to \infty$. One can show that

$$\sup_{z \in C_N} |\cot \pi z| \le 2. \tag{37.4}$$

The proof is left as an exercise. Then

$$\begin{aligned} |I_N| &\le \int_{C_N} |f(z)| \underbrace{|\cot \pi z|}_{\le 2 \text{ by } (37.4)} |dz| \\ &\overset{\text{Exercise 28.5}}{\le} \frac{M}{N^2} \int_{C_N} |dz| = \frac{4(2N+1)M}{N^2} \to 0 \quad \text{as } N \to \infty. \end{aligned}$$

Taking the limit in (37.3) as $N \to \infty$, we get

$$\lim_{N\to\infty} \sum_{n=-N}^{N} f(n) = -\pi \sum_{k=1}^{K} \operatorname{Res} f(z) \cot \pi z.$$

Under our conditions on f, the series on the left-hand side is absolutely convergent and the theorem is proven. $\square$

REMARK 37.3. *With certain modifications Theorem* 37.2 *can be extended to functions f which may have a pole at $z = 0$. Then* (37.2) *reads*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} f(n) = -\pi \sum_{k=1}^{K} \operatorname{Res}\left(f(z)\cot \pi z, z_k\right)$$

where $\{z_k\}_{k=1}^{K}$ is the set of all the poles of f(z), including 0.

Exercises

Exercise 37.1 Prove (37.4).

Exercise 37.2 Prove Remark 37.3.

Exercise 37.3 Show that

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2+1} = \pi \coth \pi.$$

Exercise 37.4 Show that

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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