Well_Known partial differential Equation

I) $U_t = \alpha\,(U_{XX} + U_{yy} + U_{zz})$

Liner three_dimensional heat equation

II) $U_{xx} + U_{yy} = 0$

Laplace equation in two dimension

III) $U_{tt} = \alpha^2\,(U_{xx} + U_{yy} + U_{zz})$

Liner three_dimensional wave equation

IV) $V_{xx} = RCV_t$, $I_{xx} = RCI_t$ Telegraph equation

V) $V_{xx} = LCV_{tt}$, $I_{XX} = LCI_{tt}$ Radio equation.

Where V=potential ,I= Current ,C=Capacitance ,L=inductance

Note: the partial differential Co_efficients are denoted as follows:

$\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$, $\frac{\partial^2 z}{\partial t^2} = t$

SOLUTION OF PARTIAL DIFFERANTIAL EQUATION:

The relation between dependent variable and independent variables

Such that : 1. This relation is empty from partial derivatives.

2. This relation satisfying this partial differential equation.

is called solution.

Kinds of solutions:

a. **General solution :**

If the solution contains arbitrary functions equal to order of this partial differential equation ,then this solution is called general solution.

b. **Particular solution**

If we choose special function and put it in general ,in this case this solution named Particular solution.

c . **Complete solution**

If the solution contains arbitrary constant only ,then this solution is called Complete solution.

Example :

1. *show that* the relation $Z = Ax^2 + By^2$ is a complete solution for p.d.e $xz_x + yz_x = 2z$ ....* ,where A,B are constants.

Solution:

$Z = Ax^2$ +By^2 $\rightarrow$ $Z_x = 2Ax$ and $Z_y = 2By$, put these in *

$$x(2Ax^2) + y(2By) = 2Ax^2 + 2By^2$$
$$= 2[Ax^2 + By^2] = 2Z$$

There for this a solution for * equation and since this solution contains only arbitrary constant , then this is complete solution.

2. *show that:*

I. The relation $Z = \emptyset(x)$ is a solution for $\frac{\partial z}{\partial y}$=0

II. The relation $Z = \emptyset_1(y) + \emptyset_2(x)$ is a solution for $\frac{\partial^2 x}{\partial x \partial y}$=0

Solution (I):

$$\frac{\partial x}{\partial y} = \frac{\partial \emptyset}{\partial u} \cdot \frac{\partial u}{\partial y} = \emptyset'(u) \cdot 0 = 0$$

$\therefore Z = \emptyset(x)$ is general solution for $\frac{\partial z}{\partial y} = 0$

Solution (II):

$$\frac{\partial x}{\partial x} = \emptyset'_1(y) \cdot 0 + \emptyset'_2(x) \cdot 1 = \emptyset'_2(x)$$
$$\frac{\partial^2 z}{\partial x \partial y} = \emptyset''_2(x) \cdot 0 = 0$$

$\therefore Z = \emptyset_1(y) + \emptyset_2(x)$ is a general solution for $\frac{\partial^2 z}{\partial x \partial y} = 0$

Example : *Verify that* : Z=f($x^2 + y^2$) is a solution of $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$___

Solution: Given $z = (x^2 + y^2)$____(1)

Differentiating (1) w.r.t x and y ,partially ,we get

$$\frac{\partial z}{\partial x} = f'(x^2 + y^2) \cdot 2x \text{ and } \frac{\partial z}{\partial y} = f'(x^2 + y^2) \cdot 2y\_\_\_\_(2)$$

Put (2) in left side of (*) ,we get right side.

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y \cdot 2xf'(x^2 + y^2) - x \cdot 2yf'(x^2 + y^2) = 0$$

Hence $Z = f'(x^2 + y^2)$ is asolution for $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$

Properties of arbitrary functions

(*) Let $\emptyset(u)$ be arbitrary function with variables u and let A be constant ,then

a. $A\emptyset(u)$ is arbitrary function.
b. $\emptyset(Au)$ is arbitrary function.
c. $\emptyset(u)+A$ is arbitrary function.
d. $\emptyset'(u)$ is arbitrary function.
e. $\int\emptyset(u)\,du$ is arbitrary function.

(**) If A,B are two arbitrary constants .then we can written one is a function to other (i.e) $A=\emptyset(B)$ or $B=\emptyset(A)$.

(***) we can written arbitrary function $\emptyset(u,v)=0$, where u,v are variables by $u=\emptyset(v)$ or $v=\emptyset(u)$.

II. Method of forming partial differential equations :

A partial equation is formed by two methods:

1. By eliminating arbitrary constants .
2. By eliminating arbitrary functions.

1.Method of elimination of arbitrary constants:

Case(1): if the number of arbitrary constants to be eliminated is less than or equal to the number of independent variables ,(i.e) $F(X,Y,Z,A,B)=0$

Then the p.d e`s obtained will be of first order.

In this case ,how to find the p.d.e?

*Differentiating partially w.r.t x.

*Differentiating partially w.r.t y.

*we get two equations ,we solve these equations by eliminating arbitrary constants ,so that we obtain p.d.e .

Case(2): If the number of arbitrary constants to be eliminated is more than the number of independent variables ,then the p.d.e`s obtained will be 2^{nd} or higher order. In this case this form is $F(X,Y,Z.A.B.C)=0$

Now how to find the p.d.e?

*Differentiating partially w.r. t x.

*Differentiating partially w.r.t y.

*To fined Z_{xx} and Z_{yy},then we get some equations and solve them, until we get of p.d.e .

ملاحظه (1) في الحالة الأولى نحصل على الأغلب على معادلة تفاضلية جزئية واحدة أو أكثر من الرتبة الأولى
(2) في الحالة الثانية نحصل على أكثر من معادلة تفاضلية جزئية ومن رتب أعلى من الرتبة الأولى

Now we shall give the following examples on the first and second case.

Example 1 : Form the partial differentialial equation be eliminating the arbitrary constants of Z=Ax^2+By^2

Solution:

$Z = Ax^2 + By^2 \to (1) \to$ the number of constant equal to independent variable

differentiating partially (1) w.r.t x, we get

$$\frac{\partial z}{\partial x} = 2xA \to A = \frac{Z_x}{2x} \to (2)$$

differentiating partially (1) w.r.t y, we get

$$\frac{\partial z}{\partial y} = 2yB \to B = \frac{Z_y}{2y} \to (3)$$

Putting (2) in(1) , we get

$$Z = \frac{Z_x}{2x}x^2 + \frac{Z_y}{2y}y^2$$

$\therefore Z = \frac{1}{2}xZ_x + \frac{1}{2}yZ_y$ p.d.e of order one

Example 2 : Form a partial differential equation from

$Z = Ax + By + Cxy \to (1)$

Solution: differentiating (1) w.r.t x, we get $Z_x = A + Cy \to (2)$

differentiating partially(1) w.r.t y, we get $Z_y = B + Cx \to (3)$

Look that : we cant eliminating the arbitrary constant ,therefore we need Z_{xx} and Z_{yy}, and Z_{xy}.

differentiating partially (2) w.r.t x ,we get $Z_{xx} = 0 \to (4)$

differentiating partially (3) w.r.t y ,we get $Z_{yy} = 0 \to (5)$

differentiating partially (2) w.r.t y, we get $Z_{xy} = c \to (6)$

Putting (6) in (2) ,we get $Z_x = A + yZ_{xy} \to A = Z_x - yZ_{xy} \to (7)$

Putting (6) in (3) , we get $Z_y = B + xZ_{xy} \to B = Z_y - xZ_{xy} \to (8)$

Putting (6) in (3) , we get $Z = x(Z_x - yZ_{xy}) + y(Z_y - xZ_{xy}) + xyZ_{xy}$

$Z = xZ_x - xyZ_{xy} + yZ_y.$

Example 3 : Form partial differential equation from : $Z = A(x+y) \to (1)$

Solution: $Z = A(x+y) \to$[the number of costant less than independent variables]

differentiating partially(1) w.r.t x, we get $Z_x = A \to (2)$

Putting (2) in(1) , we get $Z = Z_x(x+y) \to$ p.d.e of order one.

or

Putting :$Z_y = A$ in (1) ,we get $Z = Z_y(x+y) \to$ p.d.e of order one.

ملاحظه

في بعض الأمثلة يعطي وصف كلامي لعلاقة رياضية المطلوب إيجاد الوصف الرياضي لهذه العلاقة (متغيرات) مستقلة ومعتمده وثوابت) ومن ثم منها نشكل p.d.e

Example 4 : Find the differential equation of all spheres whose centers lie on the x_axis with radius equal to 1.

Solution: we know that the equation of the sphere with center(a,b,c) and radius r is given by

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2 \to (1)$

Since the center lies on x_axis $\therefore b = 0 , c = 0$,then (1) reduces to

$(x-a)^2 + y^2 + z^2 = 1 \to (2)$

The relation (2) contains the number of constant less than the independent variables.

Arbitrary constant =a , independent variables =x ,y

Differentiating partially (2) w.r t x, we get

$2(x-a).1 + 0 + 2zZ_x = 0 \to (3)$

Differentiating partially (2) w.r t y, we get $0 + 2y + 2zZ_y = 0 \to (4)$

From(2) we get $x - a = -zZ_x \to (5)$

Putting (4) in(2) , we get $(-zZ_x)^2 + y^2 + z^2 = 1$

$\therefore Z^2Z_x^2 + y^2 + Z^2 = 1$ is p.d.e of order one.

While the equation (4) $2y + 2zZ_y = 0$ is p.d.e of order one in this case we het two p.d.e

Example 5 : Find the differential equation of all spheres whose centers lie on y_axis , with radius equal to r.

Solution: this center of this spheres is(0,b,0) and radius equal=r

$x^2 + (y-b)^2 + Z^2 = r^2 \to (1)$

$2x + 2zZ_x = 0 \to (2)$ [Differentiating (1) w.r.t x]

$2(y-b) + 2zZ_y = 0 \to (3)$ [Differentiating (1) w.r.t y]

From(3) , we get $y - b = -ZZ_y \to (4)$

Put (4) in (1) we get $x^2 + (-ZZ_y)^2 + Z^2 = r^2$

$\therefore x^2 + Z^2Z_y^2 + Z^2 = r^2$ and $2x + 2ZZ_x = 0$ are p.d.e of order one.

Example 6 : Find the differential equation of spheres whose lie on the Z_axis.

Solution: the equation of sphere with center (a ,b, c) are radius is the form $(x-a)^2+(y-b)^2+(z-c)^2=r^2 \rightarrow (1)$

But the center lie on z_axis ,(i.e) the center is(0,0,c) and (1) radius

$x^2+y^2+(z-c)^2=r^2 \rightarrow (2)$

We need to eliminate c and r .

Differentiating (2) w.r.t x and y , we get

$2x+2(z-c).\frac{\partial z}{\partial x}=0 \rightarrow z-c=-x.\frac{1}{\frac{\partial z}{\partial x}} \rightarrow (3)$

And $2y+2(z-c).\frac{\partial z}{\partial y}=0 \rightarrow z-c=-y.\frac{1}{\frac{\partial z}{\partial y}} \rightarrow (4)$

By equality (3) and (4) ,we get

$\rightarrow -\frac{x}{\frac{\partial z}{\partial x}}=-\frac{y}{\frac{\partial z}{\partial y}}$

$x\frac{\partial z}{\partial y}-y\frac{\partial z}{\partial x}=0$ is the required partial differential equation

Example 7 : Find the partial differential equation of all spheres of radius 3 units having there centres in xy_plane.

Solution: the center of this sphere is (a,b,0) and this equation is

$(x-a)^2+(y-b)^2+z^2=g \rightarrow (1)$

Differentiating partially (1) w.r.t x and y , we get

$2(x-a)+2Z\frac{\partial z}{\partial y}=0 \rightarrow (x-a)=-Z\frac{\partial Z}{\partial x} \rightarrow (2)$

And

$2(y-b)+2Z\frac{\partial z}{\partial y}=0 \rightarrow (y-b)=-Z\frac{\partial z}{\partial y} \rightarrow (2)$

Putting (2) in(1) ,we get

$Z^2(\frac{\partial z}{\partial x})^2+Z^2(\frac{\partial Z}{\partial y})^2+Z^2=9$ is the required p.d.e.

2- Method of elimination of arbitrary functions.

If the partial differential equation is obtained by elimination of arbitrary functions, then the order of the partial differential equation in general, equal to the number of arbitrary functions eliminated .

Now ho to find p.d.e?

*Differentiating partially w.r.t x.

*Differentiating partially w.r.t y.

*By elimination the arbitrary function with derivative from equation by solving them ,and we get p.d.e .

Example 8 : Form partial differential equation from

$Z = f(x^2 + y^2) \to (1)$

Solution: Differentiating partially ,(1) w.r.t x and ,we get

$\frac{\partial z}{\partial x} = f'(x^2 - y^2). 2x \to (2)$

$\frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y) \to (3)$

Dividing (2) over (3) ,we get

$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{x}{-y} \to y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$ is the required p.d.e

Remark: here the given p.d.e contains one arbitrary function , therefore the required p.d.e is also of order one.

Example 9 : Form the partial differentiating equation from

$Z = f(x + ay) + g(x - ay) \to (1)$

Solution: Differentiating partially (1) w.r.t x and y , we get

$\frac{\partial z}{\partial x} = f'(x + ay) + g'(x - ay) \to (2)$

$\frac{\partial z}{\partial y} = af'(x + ay) - ag'(x - ay) \to (3)$

Again differentiating(2) w.r.t x and (3) w.r t y ,we get

$\frac{\partial^2 x}{\partial x^2} = f''(x + ay) + g''(x - ay) \to (4)$

$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x + ay) + a^2 g''(x - ay)$

$= a^2[f''(x + ay) + g''(x - ay)]$,but

$$f''(x + ay) + f''(x - ay) = \frac{\partial^2 z}{\partial x^2}$$

$\therefore \frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$ is p.d.e

Remark: here the given p.d.e contains two arbitrary function, therefore the required p.d.e will be of order 2.

Ex: Form the partial differential equation from $Z = f\left(\frac{y}{x}\right)$.

Ex: Form the partial differential equation from $Z = f_1(x). f_2(y)$

III- Equation Solvable Direct Integration

The equation containing only one partial derivative can be solve by direct integration for constants of integration , we must use arbitrary function of variable kept constant.

Example 1 : *solve* $Z_y = y(x-y)$

Solution: $\frac{\partial z}{\partial y} = y(x-y) \to \int \frac{\partial z}{\partial y}.\partial y = \int y(x-y).\partial y + \emptyset(x)$

$Z = \frac{1}{2}xy^2 - \frac{1}{3}y^3 + \emptyset(x)$ is a general solution

ملاحظه تعتبر هذه المعادلة هي معادلة تفاضلية اعتيادية من الدرجة الأولى بالمتغير التابع z والمستقل y بينما المتغير x يعتبر ثابت لعدم وجود مشتقة له.

Example 2 : *solve* $Z_{xx} - 6x = 0$

(تعتبر معادلة تفاضلية اعتيادية بمتغير تابع z ومستقل x والمتغير y هو ثابت)

Solution: $Z_{xx} = 6x$

By the first integration ,we get: $Z_x = 3x^2 + \emptyset(y)$ w.r.t x

By the second integration ,we get :

$Z = x^3 + x\emptyset_1(y) + \emptyset_2(y)$ w.r.t x

This is a general solution contains two arbitrary functions because the p.d.e of order 2.

Example 3 : *solve* $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\mathrm{xy}} \to (1)$

Solution: : as the given equation (1) contains only one partial derivative.

∴it can be solved by direct integration.

Now $(1) \to \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{1}{xy} \to (2)$

Integration both sides of(2) w.r.t x ,keeping y as constant , we get $\frac{\partial z}{\partial y} = \int \frac{1}{xy}dx +$

$f(y) = \frac{1}{y}logx + f(y) \to (3)$

Again ,integrating both sides of (3) w.r.t y ,keeping x as constant, we get

$$Z = logx \int \frac{1}{y}dy + \int f(y)dy + \emptyset(x)$$

$$Z = logx.logy + g(y) + \emptyset(x), where \int f(y)dy = g(y).$$

Is the required solution.

Ex: *solve* $\frac{\partial^2 z}{\partial y^2} = \sin(xy)$

Ex: *solve* $\log\left(\frac{\partial^2 z}{\partial x \partial y}\right) = x + y$

VI. Solution Of Linear Partial Differential Equations Of The First Order

A differential equation involving first order partial derivative p and q only ,is called a partial differential equation of first order. If p and q both occur in the fist degree only and are not multiplied together , then it is called linear partial differential equation of first order.

Lagrange's Linear Equation

Is an equation of the type Pp + Qq = R

Where P,Q,R are function of x ,y ,z and $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$

Joseph Louis Lagrange [1736_1813] , a great French mathematician spent 20 years of his life in Prussia and then returned to Paris .his important major work was in calculus of variation, celestial and general mechanic, differential equations and algebra.

To solve Lagrange's linear equation.
First step: write down the auxiliary equations.

Second step: find two independent solution u(x ,y)=a and v(x ,y)=b
Third step: the solution is $\emptyset(u, v) = 0 \; or \; u = f(r)$

Example 1 : Find the general solution of partial differential equation

$Zy^2p - Zx^2q = x^2y$

Solution: $P = y^2z \,, Q = -x^2z \,, R = x^2y$

The auxiliary system is $\frac{dx}{y^2z} = \frac{dy}{-x^2z} = \frac{dz}{x^2y}$

From $\frac{dx}{y^2z} = \frac{dy}{-x^2z}$ we obtain

$-\int x^2dx = \int y^2dy \rightarrow -\frac{x^3}{3} = \frac{y^3}{3} + c_1 \rightarrow x^3 + y^3 = -\frac{3C_1}{a}$

$\therefore u = x^3 + y^3 = a$ and

From $\frac{dy}{-x^2z} = \frac{dz}{x^2y}$, we obtain

$\int ydy = -\int zdz \rightarrow \frac{1}{2}y^2 + \frac{1}{2}z^2 = c_2 \rightarrow y^2 + z^2 = 2c_2$

This the general solution is

$\emptyset(u, v) = 0 \rightarrow \emptyset(x^3 + y^3, y^2 + z^2) = 0$

Example 2 : Find the general solution of differential equation: $xp + yq = 3z$

Solution: P=x ,Q=y , R=3z

The auxiliary system is $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$

From $\frac{dx}{x} = \frac{dy}{y} \rightarrow \int \frac{dx}{x} = \int \frac{dy}{y} \rightarrow lnx = lny + lnc$

$\therefore ln\frac{x}{y} = lnc_1 \rightarrow u = \frac{x}{y} = a \text{ and from } \frac{dx}{x} = \frac{dz}{3z}$

$\int \frac{dx}{x} = \int \frac{dz}{3z} \rightarrow 3ln = lnz + lnc_2$

$\rightarrow lnx^3 - lnz = lnc_2$

$\rightarrow ln\frac{x^3}{z} = lnc_2 \rightarrow \frac{x^3}{z} = c_2$

$\therefore v = \frac{x^3}{z} = b$

Thus the general solution is $\emptyset\left(\frac{x}{y}, \frac{x^3}{z}\right) = 0$

Example 3 : Find the solution

$$x(y-z).\frac{\partial z}{\partial x} + y(z-x)\frac{\partial z}{\partial y} = z(x-y)$$

Solution: the auxiliary system is

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \rightarrow *$$

To find u(x ,y ,z)= a?

Note since $\frac{1}{2} = \frac{2}{4} \rightarrow \frac{1+2}{2+4} = \frac{3}{6}$

$$\frac{dx + dy}{x(y-z) + y(z-x)} = \frac{\text{dz}}{z(x-y)}$$

$$\frac{dx + dy}{x(y-z) + y(z-x)} = \frac{-dz}{z(y-x)}$$

$\therefore dx + dy = -dz$

$dx + dy + dz = 0$

By integration of above equation ,we get

$u = x + y + z = a$

Now to find v(x ,y ,z)=b?

Multiply * by xyz ,we get

$$\frac{yzdx}{y-z} = \frac{xzdy}{z-x} = \frac{xydz}{x-y}$$

$$\therefore \frac{yzdx}{y-z} = \frac{xzdy}{z-x} = \frac{-xydz}{y-x}$$

$\therefore yzgx + xzdy + xydz = b \rightarrow z(ydx + xdy) + xydz = 0$
$\rightarrow zd(xy) + (xy)dz = 0 \rightarrow d(xyz) = 0 \rightarrow xyz = b$

$\therefore (x + y + z, xyz) = 0$ **is a general solution.**

Example 4 : *solve* $(x^2 - yz)p + (y^2 - zx)q = \mathrm{z}^2 - \mathrm{xy} \rightarrow (1)$

Solution: **the auxiliary equations are**

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

or $$\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy} = \frac{dz-dx}{z^2-xy-x^2+yz}$$

$$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(x+y+z)(y-z)} = \frac{dz-dx}{(x+y+z)(z-x)}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x} \rightarrow (2)$$

Integrating first two members of(2), we have

$\ln(x - y) = \ln(y - z) + lna$

$ln\frac{x-y}{y-z} = lna \rightarrow \frac{x-z}{y-z} = a$

$\therefore u(x, y, z) = \frac{x-y}{y-z} = a$

Similarly from last two members of (2) ,we have

$\ln(y - z) = \ln(z - x) + lnb \rightarrow \frac{y-z}{zz-x} = b$

$\therefore v(x, y, z) = \frac{y-z}{z-x} = b$

The required solution is $\emptyset \left[\frac{x-y}{y-z}, \frac{y-z}{z-x}\right] = 0$

Method of Multipliers:

Let the auxiliary equations be $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ l ,m ,n may be constants or functions of x ,y ,z. then we have.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$$

L ,m ,n are chosen in such a way that :

$lP+mQ+nR=0$ $Thus$ $ldx+mdy+ndz=0$

Solve the differential equation ,If the solution is u=a.

Similarly ,choose another set of multipliers (l ,m ,n) and if the second solution is v=b.

∴Required solution is $\emptyset(u,v)=0$

Example 5: Find the general solution of partial equation

$(x^2-y^2-z^2)p+2xyq=2xz \ \rightarrow(1)$

Solution: the auxiliary system is

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$$

From $\frac{dy}{2xy}=\frac{dz}{2xz}$,we obtain

$$\int\frac{dy}{y}=\int\frac{dz}{z}\rightarrow lny=lnz+lna$$

$$\therefore ln\frac{y}{z}=\ln a\ \rightarrow\frac{y}{z}=a\ \rightarrow u=\frac{y}{z}=a$$

From $\frac{xdx+ydy+zdz}{x^3-xy^2-xz^2+2xy^2+2xz^2}=\frac{dz}{2xz}$ ➡ $\frac{xdx+ydy+zdz}{x^3+xy^2+xz^2}=\frac{dz}{2xz}$

$$\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}=\frac{dz}{2xz}\rightarrow\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}=\frac{dz}{z}$$

By integration the above equation ,we get

$$\text{Ln}(x^2+y^2+z^2)=lnz+lnb\ \rightarrow ln\frac{x^2+y^2+z^2}{z}=lnb$$

$$\therefore v=\frac{x^2+y^2+z^2}{z}=b$$

Thus the general solution is $\emptyset\left(\frac{y}{z},\frac{x^2+y^2+z^2}{z}\right)=0$

Example 6 : Find the equation of the surface to $(2x + 1)q - p = 0$ which satisfying Z(1,y)= y-4

Solution: the auxiliary system is $\frac{dx}{-1} = \frac{dy}{2x+1} = \frac{dz}{0}$

$\therefore R = 0 \rightarrow Z = 0$

From $\frac{dx}{-1} = \frac{dy}{2x+1} \rightarrow \int (2x + 1)dx = \int dy$

$\therefore v = x^2 + x + y = b$

The general solution $\emptyset(z, x^2 + x + y) = 0$

Now we applied the condition:

$\therefore a = y - 4 \quad , but \;\; b = x^2 + x + y$

$b = 1 + 1 + y$

$\rightarrow b = z + y$

$\therefore a - b = y - 4 - 2 - y$

$\therefore a - b = -6$

$Z - (x^2 + x + y) = -6$

$\therefore Z = x^2 + x + y - 6$ is equation of the surface.

Ex(1): Find the solution of p.d.e :$tan\frac{\partial z}{\partial y} + tanx\frac{\partial z}{\partial x} = tanz.$

(2) solve $y^2p - xyq = x(z - 2y)$

(3) solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

Non _Linear Partial Differential Equation Of The First Order

A partial differential equation which in volves first order partial derivatives p and q with degree higher than one and the product of p and q is called a non_ linear partial differential equation of the first order.

Remark : the complete solution of non_ linear partial differential equation involves only two arbitrary constant [(i.e): equal to the number of independent variables]

Some Standard Forms

I. Equation of the form f(p ,q)=0

This equations containing p and q only.

Method : Let the required solution be

$$Z = ax + by + c \rightarrow (1)$$

$$\frac{\partial z}{\partial x} = a \;, \frac{\partial z}{\partial y} = b$$

On putting these values in f(p ,q)=0 ,we get f(a ,b)=0.

From this ,find the value of b in terms of a and substitute the value of b in(1) , that will be the required solution.

Example 1: *solve* the equation $p^2 - q^2 = 1$.

Solution: the equation of the form f(p ,q)=0 ,where a=p ,q=b $\therefore f(a,b) = 0 \rightarrow a^2 - b^2 = 1$

$\rightarrow = \mp(a^2-1)^{\frac{1}{2}}$

The solution of this form is $Z = ax + by + c$

Thus $Z = ax \mp (a^2-1)^{\frac{1}{2}} y + c$ is reuired solution

Example 2 :Find the solution of pq+p+q=0

Solution: : this is of the form f(p ,q)=0, where a=p ,b=q

$\therefore f(p,q) = pq + p + q = 0$

$f(a,b) = ab + a + b = 0 \rightarrow b \frac{-a}{a+1}$

The form of solution of $f(p,q) = 0$ is $Z = ax + by + c$

$$\therefore Z = ax - \frac{a}{a+1} y + c$$

Example 3: *solve* $p^3 - q^3 = 0 \rightarrow (1)$

Solution: equation (1) is of the form $f(p,q) = 0$. therefore the solution is

$Z = ax + by + c \quad , where\ f(a,b) = 0$

$$\therefore a^3 - b^3 = 0 \rightarrow a^3 = b^3 \rightarrow a = b$$

Hence the required solution of (1) is given by

$Z = ax + by + c$

Ex: solve the following equations :

(1) $p = e^q$ (2) $p^2 + p = q^2$ (3) $\sqrt{p} + \sqrt{q} = 1$ (4) pq=1

II. Equations of the form $Z = px + qy + f(p,q)$

The solution of above equation is

$Z = az_{by} + f(a,b)$

Example 4: *solve* $Z = px + qy - 5pq$

Solution: $f(p,q) = -5pq \rightarrow f(a,b) - 5ab$

Then $Z = ax + by + f(a,b)$

Hence $Z = ax + by - 5ab$

Example 5: *solve* $Z = px + qy + 3p^{\frac{1}{3}} q^{\frac{1}{3}}$

Solution: $f(p,q) = 3p^{\frac{1}{3}} q^{\frac{1}{3}} \to f(a,b) = 3a^{\frac{1}{3}} b^{\frac{1}{3}}$

$\therefore Z = ax + by + 3a^{\frac{1}{3}} b^{\frac{1}{3}}$

Example 6: *solve* $Z = px + qy + p^2 + pq + q^2$

Solution: $f(p,q) = p^2 + pq + q^2 \to f(a,b) = a^2 + ab + b^2$

$\therefore Z = ax + by + a^2 + ab + b^2$

Ex: solve $Z = px + qy + 2\sqrt{pq}$

Ex: solve $Z = px + qy + \sin(p+q)$

Ex: solve $(pq - p - q)(z - px - qy) = pq$

III. Equation of the form $f(z,p,q) = 0 \ \to (1)$

this equations not containing x and y.

let z be function of u where $u = (x + ay)$

$$\frac{\partial u}{\partial x} = 1 \ \ and \ \ \frac{\partial u}{\partial y} = a$$

Then $p \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} . \frac{\partial u}{\partial x} = \frac{dz}{du} \ \to (2)$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} . \frac{\partial u}{\partial y} = a\frac{dz}{du} \to (2)$$

Putting the values of p and q in(2) by(1) ,then we get

$f\left(Z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$, which is an ordinary differential equation of the first order.

Rule(1): solve the last ordinary differential equation in terms of Z and u.

(2): replace u by x+ay , we get the required solution.

Example 7: *solve* $Z = p^2 + q^2 \ \to (1)$

Solution: put $u = x + ay \ \to \frac{\partial u}{\partial x} = 1 \ , \frac{\partial u}{\partial y} = a$

Therefore $p = \frac{dz}{du}$, $q = a\frac{dz}{du} \ \to (2)$

Put (2) in (1) ,we get .

$$Z = \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2$$

$$Z = \left(\frac{dz}{dy}\right)^2 [1 + a^2]$$

$$\left(\frac{dz}{dy}\right)^2 = \frac{z}{1+a^2}$$

By taking the square root of previous statement:

$$\frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}} \quad] * du$$

$$\frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}}\, du$$

By integration of above equation: $\int \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}} \int du$

$$\therefore 2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}\, u + k$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}\, u + \frac{\sqrt{1+a^2}}{\sqrt{1+a^2}}\, k$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}\left[u + \sqrt{1+a^2}\, k\right] \quad , but\ \sqrt{1+a^2}\ \ k = b$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}\ [u+b]\, , but\ \ u = x + ay$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}\ [x+ay+b]$$

$2\sqrt{z}.\sqrt{1+a^2} = x + ay + b \ \rightarrow 4z(1+a) = (x+ay+b)^2$ *is* the required solution

Example 8: *solve* $p^2 = zq \ \rightarrow (1)$

Solution: the given equation (1) is the form $f(z,p,q) = 0.$

Therefore take $u = x + ay \rightarrow p = \frac{dz}{du}\, , q = a\frac{dz}{du} \rightarrow (2)$

Put (2) in(1) ,we get $\left(\frac{dz}{du}\right)^2 = az\left(\frac{dz}{du}\right) \rightarrow \frac{dz}{du}\left[\frac{dz}{du} - az\right] = 0$

But $\frac{dz}{du} \neq 0$

$$\therefore \frac{dz}{du} - az = 0 \rightarrow \frac{dz}{du} = az] \ \frac{du}{z} \rightarrow \frac{dz}{z} = adu \ \rightarrow (3)$$

Integration of (2) ,we get

$$lnz = au + lnb \rightarrow lnz - lnb = au \rightarrow ln\frac{z}{b} = au \rightarrow \frac{z}{b} = e^{au}$$

$z = be^{au}\ \ but\ \ u = x + ay \ \ \therefore z = be^{ax+a^2y}$ is required solution

Ex: solve $p(1+q) = qz$

Ex: solve $g(p^2z + q^2) = 4.$

IV. Equation of the form $f_1(x,p) = f_2(y,q)$

The equation in which z is absent and the terms involving x and p can be separated from those involving y and q.

Step (1): Find p and q from the given equation

Step (2): Use $dz = pdx + qdy$ and on integrating , find z in terms of x and y , as the required solution .

Example 9 : *solve* $p - x^2 = q + y^2$

Solution: this equation of the form $f_1(x,p) = f_2(y,q)$

$$p - x^2 = q + y^2 = a$$

$$\therefore p - x^2 = a \ \rightarrow p = x^2 + a \quad and \quad q + y^2 = a \rightarrow q = a - y^2$$

Putting these values on $dz = pdx + qdy$

$$\rightarrow dz = (x^2 + a)dx + (a - y^2)dy$$

$$\int dz = \int (x^2 + a)dx + \int (a - y^2)dy + c$$

$$\rightarrow Z = ax + \frac{1}{3}x^3 + ay - \frac{1}{3}y^3 + c$$

$\therefore Z = a(x + y) + \frac{1}{3}(x^3 - y^3) + c$ is the required solution

Example 10 : *solve* $q = xyp^2 \rightarrow (1)$

Solution: $q = xyp^2] * y^{-1} \rightarrow p^2x = qy^{-1} = a$

$$\rightarrow p^2x = a \ \rightarrow p = \sqrt{\frac{a}{x}} \quad and \ qy^{-1} = a \ \rightarrow q = ay$$

Putting the values of pand q in $dz = pdx + qdy$

$$dz = \sqrt{a}\, x^{\frac{-1}{2}} + aydy$$

Integration $\int dz = \int \sqrt{a}\ x^{\frac{-1}{2}} + \int aydy + c$

$\therefore Z = 2\sqrt{ax} + ay + c$ *is* the required solution.

Ex: *solve* $q(p - cosx) = cosy$

Ex: *solve* $yp + xq + pq = 0$

Ex: *solve* $\sqrt{p} + \sqrt{q} = x + y$

Ex: *solve* $pe^y = qe^x$

Differential Equation

Ordered differential equation | partial differential equation

Partial Differential Equation

Def(1): A partial differential equation are those equations which contain partial differential coefficients , independent variables and dependent variables the independent variables will be denoted by x and y and the dependent variables by z.

Remark: (1) If the dependent variable z is a function of two independent variables (i.e) $z = (x, y)$, then the form of p.d.e is

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \ldots\right) = 0$$

(2) If the dependent variable u is a function of three independent variables (i.e) $u = u(x, y, z)$,then the form of p.d.e is

$$\boldsymbol{F(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xz}, \ldots) = 0}$$

Def(2): the order of a partial differential equation is the order of the highest partial derivative occurring .

Def(3): the degree of a partial differential equation is the degree of the highest order partial derivative occurring in the equation

Def(4): A differential equation involving first order partial derivative $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ only is called a partial equation of the first order.

If $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ both occur in first degree only and are not multiplied together , then it is called a linear partial differential equation of first order.

Def(5): A linear partial differential equation is homogeneous if all the terms contain derivatives of the same order.

Note : the partial differential equation is classification by

1.order 2.degree 3.linearly 4.homogenous

Q: Classify the following p.d.e by order ,degree, linearly and homogenous:

	Equations	order	degree	linearly	homogenous
1	$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0$	1	1	✓	✓
2	$Z\left(x^2-z_y+y^2z_x\right)=xy^2$	1	1	non	✓
3	$z_{xx}^2+2z_{xy}+3z_y^2+xy^2z=0$	2	2	non	Non
4	$\frac{\partial z}{\partial x}+xz=y$	1	1	✓	✓
5	$\frac{\partial z}{\partial y}=2y(\frac{\partial z}{\partial x})^2$	1	2	non	✓
6	$(\frac{\partial^3 z}{\partial x^3})^2+\frac{2}{3}\frac{\partial^2 z}{\partial x\,\partial y}+3(\frac{\partial z}{\partial y})^3+xy^2z=0$	3	2	non	Non
7	$yu_y-zx^3y^3u_{xy}=g(x,y)$	2	1	✓	Non
8	$x^2Z_{xx}-Z^2=0$	2	1	non	✓
9	$Z_x\ln Z_y=2Z^3$	1	1	non	Non
10	$ZZ_x-\sin(x+y)\,Z_y=xcosz$	1	1	non	✓

ملاحظه : تسمى المعادلة التفاضلية الجزئية بالخطية اذا كان مجموع قوى Z ومشتقاتها في أي حد من حدودها تساوي واحد بشرط أن تكون تلك القوة عدد صحيح غير سالب .