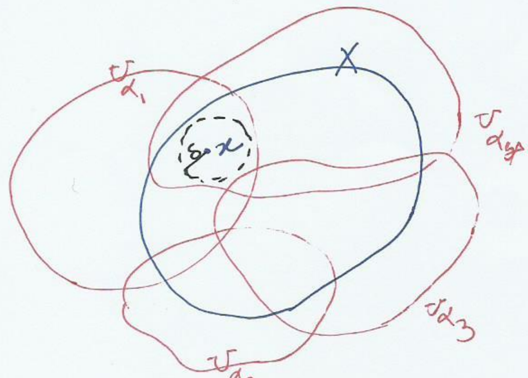
Lebesgue Covering Lemma:

If $\{U_\alpha\}$ is an open cover of compact X, then $\exists \delta > 0$ s.t. $\forall x \in X$ $B(x,\delta)$ is contained in some U_α. (δ is called Lebesgue number of the cover).

$B(x,\delta) \subseteq U_{\alpha_1}$
$B(x,\delta) \subset U_{\alpha_4}$



proof: Since X is compact, $\exists$ finite subcover $\{U_{\alpha_i}\}_{i=1}^{n}$. If K is closed, define $d(x,K) = \inf\{d(x,y) : y \in K\}$.

claim: $d(x,K)$ is continuous function of x then $f(x) = \frac{1}{n}\sum d(x, U_{\alpha_i}^c)$ continuous, $f(x)$ is defined on compact set. so, it attains min. and max value. we call the min value as δ so if $f(x) \geq \delta$ then at Least on of $d(x, U_{\alpha_i}^c) \geq \delta$ so for this i $B(x,\delta) \subset U_{\alpha_i}$ ∎

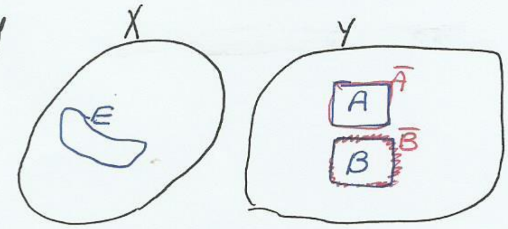
Theorem: $f: X \to Y$ continuous, E connected, then $f(E)$ is connected.

proof: by contradiction, suppose $f(E)$ is not connected, then $f(E) = A \cup B$ is a separation $(\bar{A} \cap B = A \cap \bar{B} = \phi)$. Notice $\begin{cases} K_A = f^{-1}(\bar{A}) \\ K_B = f^{-1}(\bar{B}) \end{cases}$ are closed (because f continuous). Let

$$\begin{cases} E_1 = f^{-1}(A) \cap E \\ E_2 = f^{-1}(B) \cap E \end{cases} \Rightarrow \text{disjoint and non-empty (because } A, B \text{ disjoint)}$$

so claim they separation of E. Notice, $\begin{cases} E_1 \subset K_A \\ E_2 \subset K_B \end{cases}$ are closed

so $\begin{cases} \bar{E}_1 \subset K_A \\ \bar{E}_2 \subset K_B \end{cases}$ & $\begin{cases} K_A \cap E_2 = \varphi : (\text{because } K_A = f^{-1}(\bar{A}),\ E_2 = f^{-1}(B),\ \bar{A} \cap B = \varphi) \\ K_B \cap E_1 = \varphi : \text{similarly} \end{cases}$

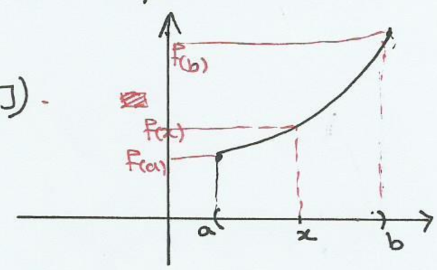


So E is separated ✗ ∎

Theorem: Intermidate value theorem:

If $f:[a,b] \to \mathbb{R}$ continuous and $f(a) < c < f(b)$ then $\exists x \in (a,b)$ s.t $f(x) = c$

proof: $[a,b]$ is connected $\Rightarrow f([a,b])$ is connected $\Rightarrow$ but $f(x)$ not achieved then c would disconnected of $f([a,b])$. ∎

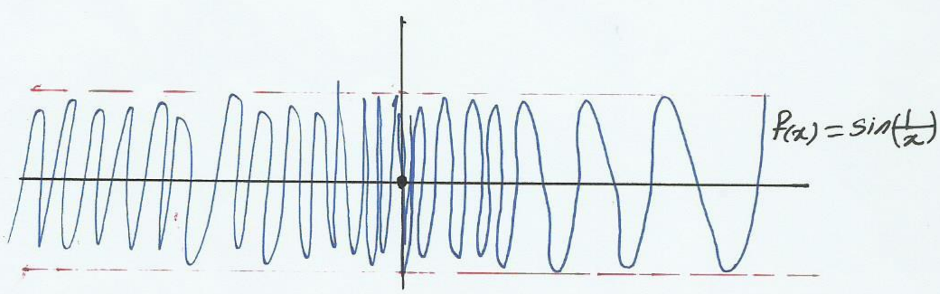


Note: Converse is false:

$$f(x) = \begin{cases} 0 & : x = 0 \\ \sin(\frac{1}{x}) & : x \neq 0 \end{cases}$$

. topologist's sine curve.

not continuous at 0 but has the intermediate value property.



Discontinuous functions:

Example: Dirichlet's function: $f(x) = \begin{cases} 1 & : x \in \mathbb{Q} \\ 0 & : x \notin \mathbb{Q} \end{cases}$

f is not continuous at any p.

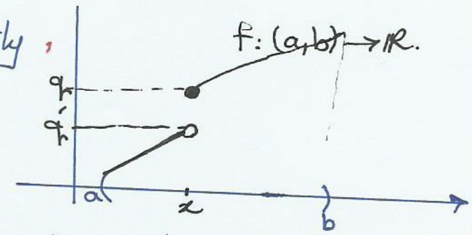
Example: $f(x)=\begin{cases} \frac{1}{q} & : x=\frac{p}{q} \\ 0 & : x\notin\mathbb{Q} \end{cases}$

discontinuous at all rational numbers, but continuous at irrational.

Note: For all $\{t_n\}$ in (a,b) $t_n \to x$ then $f(t_n)\to q$ and write

$f(x^+)=q$ or $\lim\limits_{t\to x^+} f(t)=q$. similarly,

$f(x^-)=q'$ or $\lim\limits_{t\to x^-} f(t)=q'$



(*): If f is discontinuous but if $f(x^+)$, $f(x^-)$, then we say f has discontinuity of the first kind.

Example: second kind at $x=0$ $\quad f(x)=\begin{cases} 0 & : x\leq 0 \\ \sin\frac{1}{x} & : x>0 \end{cases}$

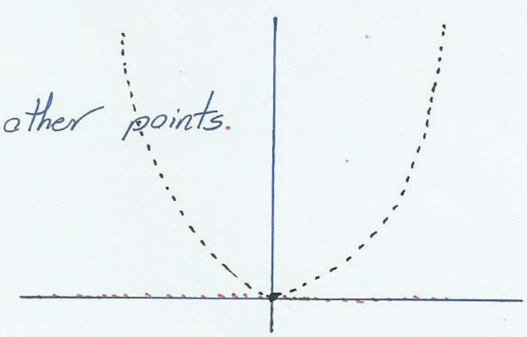
limit is not exist.

Example: on dirichlet 's function discontinuity of second kind.

Example: $f(x)=\begin{cases} \frac{1}{q} & : x=\frac{p}{q} \\ 0 & : x\notin\mathbb{Q} \end{cases}$. first kind discontinuity.

Example: $f(x)=\begin{cases} x^2 & : x\in\mathbb{Q} \\ 0 & : x\notin\mathbb{Q} \end{cases}$ is continuous at zero, and

has discontinuity of second kind for other points.



Monotonic functions: $f:(a,b)\to\mathbb{R}$

f monotonic increasing when if $x\leq y \Rightarrow f(x)\leq f(y)$

f monotonic decreasing when if $x\leq y \Rightarrow f(x)\geq f(y)$

Theorem: If f monotonic increasing in (a,b) then $f(x^+)$, $f(x^-)$ exist for $\forall x\in(a,b)$.

proof. In fact, $\sup\limits_{t\in(a,b)}(f(t)) \leq f(x) \leq \inf\limits_{t\in(a,b)}(f(t))$ - because,

claim, $A=f(x^-)$

Given $\varepsilon>0$ consider $A-\varepsilon$ so by defenition of sup

$\exists\,\delta>0$ s.t. $A-\varepsilon < f(x-\delta)\leq A \Rightarrow$ (because A is sup).

but then any $t\in(x-\delta, x)$ but satisfy $f(x-\varepsilon)\leq f(x)\leq f(A)$

so $f(t)\in(A-\varepsilon, A)$ as desired. similarly, argument on the other side. ■

Corollary: Monotonic functions have the discontinuity of 2nd kind.

Theorem: f monotonic on (a,b) set of points where f is continuous is countable.

proof: $\underbrace{\forall x \text{ where } f \text{ is discontinuous}}_{D}$ pick $r(x)\in\mathbb{Q}$ st. $f(x^-)\leq f(x)\leq f(x^+)$

if $x,y\in D$ $r(x)\neq r(y) \Rightarrow$ because f is monotone get 1-1 corresponds and subset of $\mathbb{Q}$ ■