$S \subset G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_{k-1}} \subset \bigcup_{i \in I} G_i$ $\therefore$ S is a compact set.

Example: Is $\mathbb{R}$ a compact set.

solution: Homework.

A set K is bdd if $K \subset B(x,r)$ for some $x \in X$.

Theorem: A compact set is always bounded.

proof: suppose that k is compact and it is not bounded. Then,

$\forall x, \forall r$, $K \not\subseteq B(x,r)$. Consider, the family $\{B(x,n)\}_{n \in \mathbb{N}}$.

This is an open covering of k. so, for any finite subcovering

$\{B(x,k_i)\}_{i=1,2,\dots,n}$?

$\bigcup_{i=1}^{n} B(x,k_i) = B(x,k_n) \not\supseteq K$.

then, k is not compact. $\times$ which is contradiction. So, k is bounded.

Theorem: A compact set is closed.

proof: Suppose that K is compact set. and $p \in K^c$ ($X-K$) and consider

$q \in K$. If consider $= \frac{d(x,y)}{2} = \frac{r}{2}$ so,

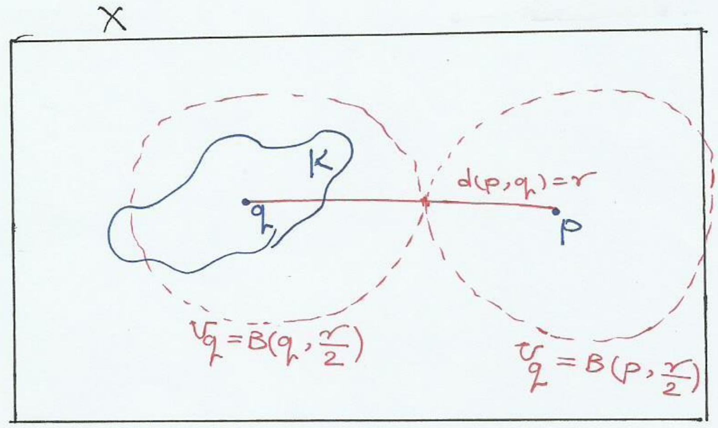
$\forall q \in K$ Let $V_q = B(q, \frac{r}{2})$, $V_q = B(p, \frac{r}{2})$ where $r = d(p,q)$

Notice $\{V_q\}$ is a cover of K. So, by compactness of k

$\exists$ finite subcover $\{V_{q_1}, V_{q_2}, \dots, V_{q_n}\}$ and $W = V_{q_1} \cap V_{q_2} \cap \dots \cap V_{q_n}$

is open ball of radius $\min d(p,q_i)$.

we claim $W \cap V_{q_i} = \phi$. (because: $W \subset U_{q_i}$ and $U_{q_i} \cap V_{q_i} = \phi$)



Theorem: ~~If~~ suppose: $Y \subseteq X$ K is compact in $Y \Longleftrightarrow K$ is compact in X

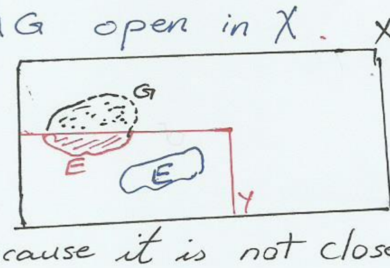
proof: H.W.

Recall:

Definition A Set U is open in Y. (relative to Y) if every point of U is an interior point of U (using metric in Y).

Theorem: $E \subset Y \subset X$

E open in $Y \Longleftrightarrow E = Y \cap G$ open in X.



proof: H.W.

Example: $(0,1)$ is not compact, because it is not closed.

$\mathbb{R}$ is not compact, because it is not bounded through it is

closed. So, closed set not necessary compact.

Theorem: A closed subset (B) of compact set (k) is compact.

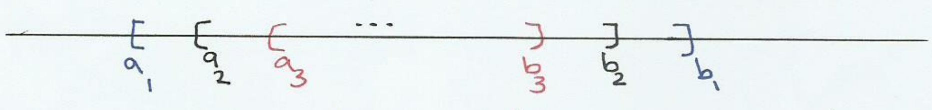
proof: Let $\{U_\alpha\}$ be an open cover of B. Notice, B^c is open because B is closed. So, $\{U_\alpha\} \cup \{B^c\}$ is an open cover of K. Therefor by compactness, $\exists$ finite subcover of $\{U_\alpha\} \cup \{B^c\}$ $\{U_{\alpha_1}, \dots, U_{\alpha_n}, B^c\}$ and, Notice: $B^c \cap B = \phi$. So, $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ cover B, and its finite cover of originial cover.

corollary: F closed, k compact then $F \cap k$ is compact.

proof: H.W. K compact $\rightarrow$ K closed & F closed $\Rightarrow F \cap K$ closed
$F \cap K \subseteq k \Rightarrow F \cap K$ compact.

Nested closed intervals: $I_n = [a_n, b_n]$

if $m > n$ then $a_n \leq a_m \leq b_m \leq b_n$



Theorem: $[a,b]$ is compact. (in $\mathbb{R}$). (K-cells is compact in $\mathbb{R}^k$).

proof: H.W.

Hein Borel theorem: In $\mathbb{R}$

K is compact $\Longleftrightarrow$ K is closed and bounded.

proof: $\Rightarrow$ Already.

$\Leftarrow$): K is bounded. so, $K \subset [-r, r]$. (For some $r > 0$).

K is closed and $[-r, r]$ is compact. so, K is compact.

Theorem: K is compact $\Longleftrightarrow$ Every infinite subset of K has a limit point in K.

proof: H.W.

Bolzano – Wirestrass theorem:

Every bounded infinite subset of $\mathbb{R}^n$ has a limit point.

proof: If the subset E is bounded then E is subset of some compact k-cell. so, by previous theorem E has a limit point in $\mathbb{R}^n$.

Example: $\mathbb{R}$ is uncountable.

solution: by contradiction suppose $\mathbb{R} = \{x_1, x_2, \dots\}$ countable

choose I_1 misses x_1 and

$I_2 \subset I_1$ misses x_1, x_2

$I_3 \subset I_2 \subset I_1$ misses x_1, x_2, x_3

Nested sequences $\Rightarrow \exists x \in \bigcap I_n$, x is not in list.